Boosting Frank-Wolfe by Chasing Gradients

Cyrille W. Combettes

with Sebastian Pokutta

School of Industrial and Systems Engineering
Georgia Institute of Technology
Atlanta, GA, USA

37$^{\text{th}}$ International Conference on Machine Learning
July 12–18, 2020

International Conference On Machine Learning

Outline

1. Introduction
2. The Frank-Wolfe algorithm
3. Boosting Frank-Wolfe
4. Computational experiments

Introduction

Let $\mathcal{H}$ be a Euclidean space (e.g., $\mathbb{R}^n$ or $\mathbb{R}^{m\times n}$) and consider

$$\begin{aligned} &\min f(x) \\ &\text{ s.t. } x \in \mathcal{C} \end{aligned}$$

where

- $f : \mathcal{H} \to \mathbb{R}$ is a smooth convex function
- $\mathcal{C} \subset \mathcal{H}$ is a compact convex set, $\mathcal{C} = \text{conv}(\mathcal{V})$

Introduction

Let $\mathcal{H}$ be a Euclidean space (e.g., $\mathbb{R}^n$ or $\mathbb{R}^{m\times n}$) and consider

$$\begin{aligned} &\min\ f(x) \\ &\ \text{s.t. } x \in \mathcal{C} \end{aligned}$$

where

- $f : \mathcal{H} \to \mathbb{R}$ is a smooth convex function
- $\mathcal{C} \subset \mathcal{H}$ is a compact convex set, $\mathcal{C} = \mathrm{conv}(\mathcal{V})$

Example

- Sparse logistic regression

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} \frac{1}{m}\sum_{i=1}^{m} \ln(1+\exp(-y_i a_i^\top x)) \\ &\text{s.t. } \|x\|_1 \leqslant \tau \end{aligned}$$

- Low-rank matrix completion

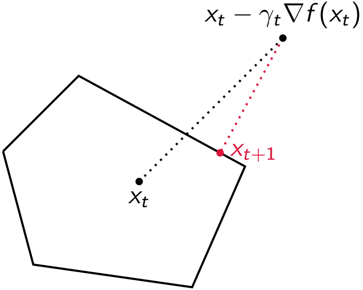
$$\begin{aligned} &\min_{X\in\mathbb{R}^{m\times n}} \frac{1}{2|\mathcal{I}|}\sum_{(i,j)\in\mathcal{I}} (Y_{i,j}-X_{i,j})^2 \\ &\text{s.t. } \|X\|_{\text{nuc}} \leqslant \tau \end{aligned}$$

Introduction

- A natural approach is to use any efficient method and add projections back onto $\mathcal{C}$ to ensure feasibility

Introduction

- A natural approach is to use any efficient method and add projections back onto $\mathcal{C}$ to ensure feasibility



Introduction

- A natural approach is to use any efficient method and add projections back onto $\mathcal{C}$ to ensure feasibility
- However, in many situations projections onto $\mathcal{C}$ are very expensive

Introduction

- A natural approach is to use any efficient method and add projections back onto $\mathcal{C}$ to ensure feasibility
- However, in many situations projections onto $\mathcal{C}$ are very expensive
- This is an issue with the method of projections, not necessarily with the geometry of $\mathcal{C}$: linear minimizations over $\mathcal{C}$ can still be relatively cheap

Introduction

- A natural approach is to use any efficient method and add projections back onto $\mathcal{C}$ to ensure feasibility
- However, in many situations projections onto $\mathcal{C}$ are very expensive
- This is an issue with the method of projections, not necessarily with the geometry of $\mathcal{C}$: linear minimizations over $\mathcal{C}$ can still be relatively cheap

Feasible region $\mathcal{C}$	**Linear minimization**	**Projection**
$\ell_1/\ell_2/\ell_\infty$-ball	$\mathcal{O}(n)$	$\mathcal{O}(n)$
ℓ_p-ball, $p \in]1,\infty[\,\setminus\{2\}$	$\mathcal{O}(n)$	N/A
Nuclear norm-ball	$\mathcal{O}(\mathrm{nnz})$	$\mathcal{O}(mn\min\{m,n\})$
Flow polytope	$\mathcal{O}(n)$	$\mathcal{O}(n^{3.5})$
Birkhoff polytope	$\mathcal{O}(n^3)$	N/A
Matroid polytope	$\mathcal{O}(n\ln(n))$	$\mathcal{O}(\mathrm{poly}(n))$

N/A: no closed-form exists and solution must be computed via nontrivial optimization

- A natural approach is to use any efficient method and add projections back onto $\mathcal{C}$ to ensure feasibility
- However, in many situations projections onto $\mathcal{C}$ are very expensive
- This is an issue with the method of projections, not necessarily with the geometry of $\mathcal{C}$: linear minimizations over $\mathcal{C}$ can still be relatively cheap

Feasible region $\mathcal{C}$	**Linear minimization**	**Projection**
$\ell_1/\ell_2/\ell_\infty$-ball	$\mathcal{O}(n)$	$\mathcal{O}(n)$
ℓ_p-ball, $p \in]1, \infty[\setminus \{2\}$	$\mathcal{O}(n)$	N/A
Nuclear norm-ball	$\mathcal{O}(\mathrm{nnz})$	$\mathcal{O}(mn \min\{m, n\})$
Flow polytope	$\mathcal{O}(n)$	$\mathcal{O}(n^{3.5})$
Birkhoff polytope	$\mathcal{O}(n^3)$	N/A
Matroid polytope	$\mathcal{O}(n \ln(n))$	$\mathcal{O}(\mathrm{poly}(n))$

N/A: no closed-form exists and solution must be computed via nontrivial optimization

Introduction

- A natural approach is to use any efficient method and add projections back onto $\mathcal{C}$ to ensure feasibility
- However, in many situations projections onto $\mathcal{C}$ are very expensive
- This is an issue with the method of projections, not necessarily with the geometry of $\mathcal{C}$: linear minimizations over $\mathcal{C}$ can still be relatively cheap

Feasible region $\mathcal{C}$	**Linear minimization**	**Projection**
$\ell_1/\ell_2/\ell_\infty$-ball	$\mathcal{O}(n)$	$\mathcal{O}(n)$
ℓ_p-ball, $p \in]1,\infty[\setminus\{2\}$	$\mathcal{O}(n)$	N/A
Nuclear norm-ball	$\mathcal{O}(\mathrm{nnz})$	$\mathcal{O}(mn\min\{m,n\})$
Flow polytope	$\mathcal{O}(n)$	$\mathcal{O}(n^{3.5})$
Birkhoff polytope	$\mathcal{O}(n^3)$	N/A
Matroid polytope	$\mathcal{O}(n\ln(n))$	$\mathcal{O}(\mathrm{poly}(n))$

N/A: no closed-form exists and solution must be computed via nontrivial optimization

Introduction

- A natural approach is to use any efficient method and add projections back onto $\mathcal{C}$ to ensure feasibility
- However, in many situations projections onto $\mathcal{C}$ are very expensive
- This is an issue with the method of projections, not necessarily with the geometry of $\mathcal{C}$: linear minimizations over $\mathcal{C}$ can still be relatively cheap

Feasible region $\mathcal{C}$	**Linear minimization**	**Projection**
$\ell_1/\ell_2/\ell_\infty$-ball	$\mathcal{O}(n)$	$\mathcal{O}(n)$
ℓ_p-ball, $p \in]1,\infty[\setminus\{2\}$	$\mathcal{O}(n)$	N/A
Nuclear norm-ball	$\mathcal{O}(\mathrm{nnz})$	$\mathcal{O}(mn\min\{m,n\})$
Flow polytope	$\mathcal{O}(n)$	$\mathcal{O}(n^{3.5})$
Birkhoff polytope	$\mathcal{O}(n^3)$	N/A
Matroid polytope	$\mathcal{O}(n\ln(n))$	$\mathcal{O}(\mathrm{poly}(n))$

N/A: no closed-form exists and solution must be computed via nontrivial optimization

Introduction

- A natural approach is to use any efficient method and add projections back onto $\mathcal{C}$ to ensure feasibility
- However, in many situations projections onto $\mathcal{C}$ are very expensive
- This is an issue with the method of projections, not necessarily with the geometry of $\mathcal{C}$: linear minimizations over $\mathcal{C}$ can still be relatively cheap

Feasible region $\mathcal{C}$	**Linear minimization**	**Projection**
$\ell_1/\ell_2/\ell_\infty$-ball	$\mathcal{O}(n)$	$\mathcal{O}(n)$
ℓ_p-ball, $p \in]1,\infty[\setminus\{2\}$	$\mathcal{O}(n)$	N/A
Nuclear norm-ball	$\mathcal{O}(\mathrm{nnz})$	$\mathcal{O}(mn\min\{m,n\})$
Flow polytope	$\mathcal{O}(n)$	$\mathcal{O}(n^{3.5})$
Birkhoff polytope	$\mathcal{O}(n^3)$	N/A
Matroid polytope	$\mathcal{O}(n\ln(n))$	$\mathcal{O}(\mathrm{poly}(n))$

N/A: no closed-form exists and solution must be computed via nontrivial optimization

- Can we avoid projections?

The Frank-Wolfe algorithm

The Frank-Wolfe algorithm (Frank & Wolfe, 1956) a.k.a. conditional gradient algorithm (Levitin & Polyak, 1966):

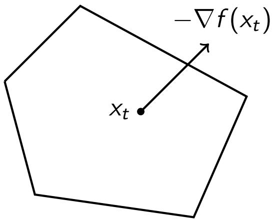
Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0,1]$.

1: **for** $t = 0$ **to** $T-1$ **do**

2: $\quad v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$



The Frank-Wolfe algorithm

The Frank-Wolfe algorithm (Frank & Wolfe, 1956) a.k.a. conditional gradient algorithm (Levitin & Polyak, 1966):

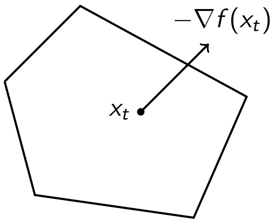
Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0, 1]$.

1: **for** $t = 0$ **to** $T - 1$ **do**

2: $\quad v_t \leftarrow \operatorname*{arg\,min}_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$



The Frank-Wolfe algorithm

The Frank-Wolfe algorithm (Frank & Wolfe, 1956) a.k.a. conditional gradient algorithm (Levitin & Polyak, 1966):

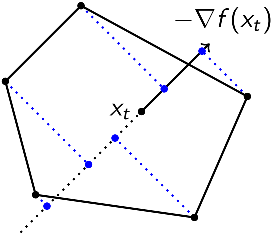
Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0,1]$.

1: **for** $t = 0$ **to** $T-1$ **do**

2: $\quad v_t \leftarrow \underset{v \in \mathcal{V}}{\arg\min} \langle \nabla f(x_t), v \rangle$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$



The Frank-Wolfe algorithm (Frank & Wolfe, 1956) a.k.a. conditional gradient algorithm (Levitin & Polyak, 1966):

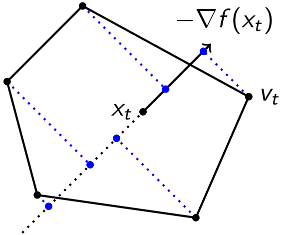
Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0, 1]$.

1: **for** $t = 0$ **to** $T - 1$ **do**

2: $\quad v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$



The Frank-Wolfe algorithm

The Frank-Wolfe algorithm (Frank & Wolfe, 1956) a.k.a. conditional gradient algorithm (Levitin & Polyak, 1966):

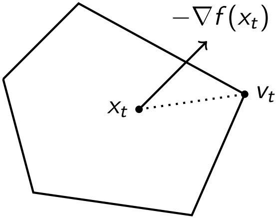
Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0, 1]$.

for $t = 0$ **to** $T - 1$ **do**

$\quad v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

$\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$



The Frank-Wolfe algorithm

The Frank-Wolfe algorithm (Frank & Wolfe, 1956) a.k.a. conditional gradient algorithm (Levitin & Polyak, 1966):

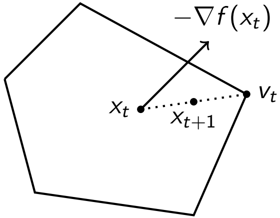
Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0, 1]$.

1: **for** $t = 0$ **to** $T - 1$ **do**

2: $\quad v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$



The Frank-Wolfe algorithm

The Frank-Wolfe algorithm (Frank & Wolfe, 1956) a.k.a. conditional gradient algorithm (Levitin & Polyak, 1966):

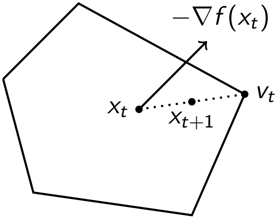
Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0, 1]$.

1: **for** $t = 0$ **to** $T - 1$ **do**

2: $\quad v_t \leftarrow \operatorname*{arg\,min}_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$



- x_{t+1} is obtained by convex combination of $x_t \in \mathcal{C}$ and $v_t \in \mathcal{C}$, thus $x_{t+1} \in \mathcal{C}$

The Frank-Wolfe algorithm

The Frank-Wolfe algorithm (Frank & Wolfe, 1956) a.k.a. conditional gradient algorithm (Levitin & Polyak, 1966):

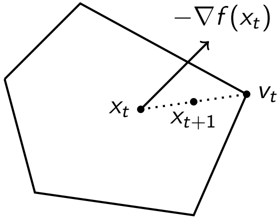
Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0, 1]$.

1: **for** $t = 0$ **to** $T - 1$ **do**

2: $\quad v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$



- x_{t+1} is obtained by convex combination of $x_t \in \mathcal{C}$ and $v_t \in \mathcal{C}$, thus $x_{t+1} \in \mathcal{C}$
- FW uses linear minimizations (the "FW oracle") instead of projections

The Frank-Wolfe algorithm

The Frank-Wolfe algorithm (Frank & Wolfe, 1956) a.k.a. conditional gradient algorithm (Levitin & Polyak, 1966):

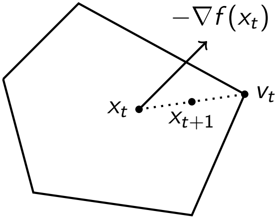
Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0, 1]$.

for $t = 0$ **to** $T - 1$ **do**

$\quad v_t \leftarrow \underset{v \in \mathcal{V}}{\arg\min} \langle \nabla f(x_t), v \rangle$

$\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$



- x_{t+1} is obtained by convex combination of $x_t \in \mathcal{C}$ and $v_t \in \mathcal{C}$, thus $x_{t+1} \in \mathcal{C}$
- FW uses linear minimizations (the "FW oracle") instead of projections
- FW = pick a vertex (using gradient information) and move in that direction

The Frank-Wolfe algorithm

The Frank-Wolfe algorithm (Frank & Wolfe, 1956) a.k.a. conditional gradient algorithm (Levitin & Polyak, 1966):

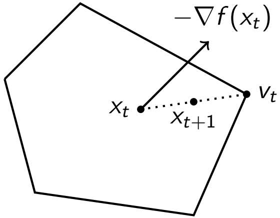
Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0, 1]$.

1: **for** $t = 0$ **to** $T - 1$ **do**

2: $\quad v_t \leftarrow \underset{v \in \mathcal{V}}{\arg\min} \langle \nabla f(x_t), v \rangle$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$



- x_{t+1} is obtained by convex combination of $x_t \in \mathcal{C}$ and $v_t \in \mathcal{C}$, thus $x_{t+1} \in \mathcal{C}$
- FW uses linear minimizations (the "FW oracle") instead of projections
- FW = pick a vertex (using gradient information) and move in that direction
- Successfully applied to: traffic assignment, computer vision, optimal transport, adversarial learning, etc.

The Frank-Wolfe algorithm

Theorem (Levitin & Polyak, 1966; Jaggi, 2013)

Let $\mathcal{C} \subset \mathcal{H}$ be a compact convex set with diameter D and $f : \mathcal{H} \to \mathbb{R}$ be a L-smooth convex function, and let $x_0 \in \arg\min_{v \in \mathcal{V}} \langle \nabla f(y), v \rangle$ for some $y \in \mathcal{C}$. If $\gamma_t = \frac{2}{t+2}$ (default) or $\gamma_t = \min\left\{ \frac{\langle \nabla f(x_t), x_t - v_t \rangle}{L\|x_t - v_t\|^2}, 1 \right\}$ ("short step"), then

$$f(x_t) - \min_{\mathcal{C}} f \leqslant \frac{4LD^2}{t+2}$$

The Frank-Wolfe algorithm

Theorem (Levitin & Polyak, 1966; Jaggi, 2013)

Let $\mathcal{C} \subset \mathcal{H}$ be a compact convex set with diameter D and $f : \mathcal{H} \to \mathbb{R}$ be a L-smooth convex function, and let $x_0 \in \arg\min_{v \in \mathcal{V}} \langle \nabla f(y), v \rangle$ for some $y \in \mathcal{C}$. If $\gamma_t = \frac{2}{t+2}$ (default) or $\gamma_t = \min\left\{\frac{\langle \nabla f(x_t), x_t - v_t \rangle}{L\|x_t - v_t\|^2}, 1\right\}$ ("short step"), then

$$f(x_t) - \min_{\mathcal{C}} f \leqslant \frac{4LD^2}{t+2}$$

- The convergence rate cannot be improved (Canon & Cullum, 1968; Jaggi, 2013; Lan, 2013)

The Frank-Wolfe algorithm

Theorem (Levitin & Polyak, 1966; Jaggi, 2013)

Let $\mathcal{C} \subset \mathcal{H}$ be a compact convex set with diameter D and $f : \mathcal{H} \to \mathbb{R}$ be a L-smooth convex function, and let $x_0 \in \arg\min_{v \in \mathcal{V}} \langle \nabla f(y), v \rangle$ for some $y \in \mathcal{C}$. If $\gamma_t = \frac{2}{t+2}$ (default) or $\gamma_t = \min\left\{ \frac{\langle \nabla f(x_t), x_t - v_t \rangle}{L \| x_t - v_t \|^2}, 1 \right\}$ ("short step"), then

$$f(x_t) - \min_{\mathcal{C}} f \leqslant \frac{4LD^2}{t+2}$$

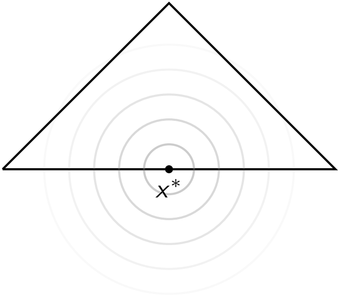
- The convergence rate cannot be improved (Canon & Cullum, 1968; Jaggi, 2013; Lan, 2013)
- Why?

The Frank-Wolfe algorithm

Consider the simple problem

$$\min \frac{1}{2}\|x\|_2^2$$
$$\text{s.t. } x \in \operatorname{conv}\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$$

and $x^* = \begin{pmatrix}0\\0\end{pmatrix}$



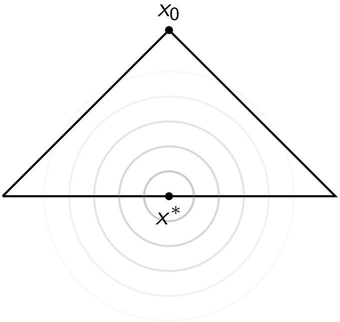
The Frank-Wolfe algorithm

Consider the simple problem

$$\min \frac{1}{2}\|x\|_2^2$$
$$\text{s.t. } x \in \operatorname{conv}\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$$

and $x^* = \begin{pmatrix}0\\0\end{pmatrix}$

- Let $x_0 = \begin{pmatrix}0\\1\end{pmatrix}$

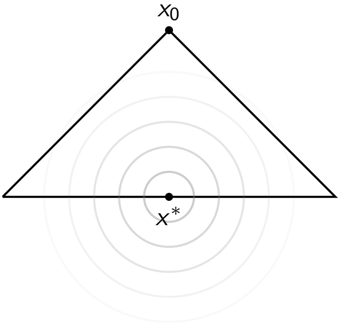


The Frank-Wolfe algorithm

Consider the simple problem

$$\min \frac{1}{2}\|x\|_2^2$$
$$\text{s.t. } x \in \operatorname{conv}\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$$

and $x^* = \begin{pmatrix}0\\0\end{pmatrix}$



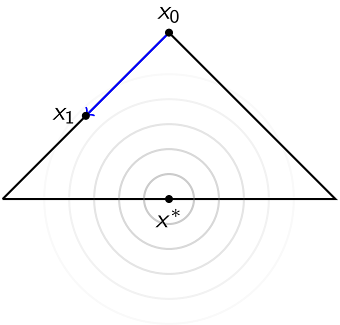
- Let $x_0 = \begin{pmatrix}0\\1\end{pmatrix}$
- FW tries to reach x^* by moving towards vertices

The Frank-Wolfe algorithm

Consider the simple problem

$$\min \frac{1}{2}\|x\|_2^2$$
$$\text{s.t. } x \in \operatorname{conv}\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$$

and $x^* = \begin{pmatrix}0\\0\end{pmatrix}$



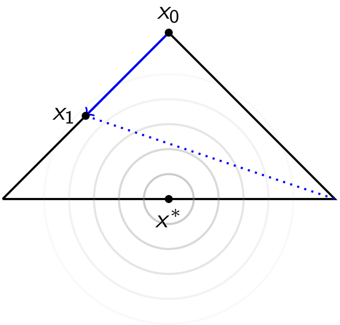
- Let $x_0 = \begin{pmatrix}0\\1\end{pmatrix}$
- FW tries to reach x^* by moving towards vertices

The Frank-Wolfe algorithm

Consider the simple problem

$$\min \frac{1}{2}\|x\|_2^2$$
$$\text{s.t. } x \in \operatorname{conv}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

and $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



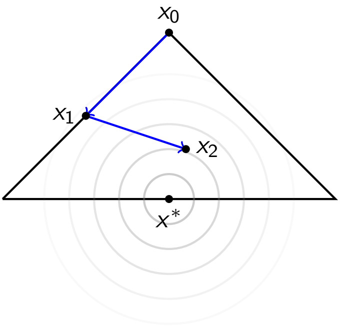
- Let $x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- FW tries to reach x^* by moving towards vertices

The Frank-Wolfe algorithm

Consider the simple problem

$$\min \frac{1}{2}\|x\|_2^2$$
$$\text{s.t. } x \in \operatorname{conv}\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$$

and $x^* = \begin{pmatrix}0\\0\end{pmatrix}$



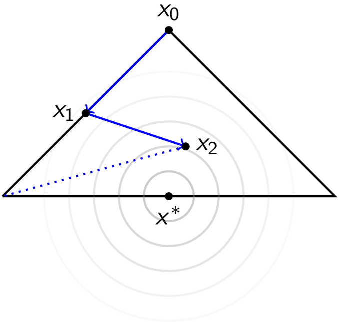
- Let $x_0 = \begin{pmatrix}0\\1\end{pmatrix}$
- FW tries to reach x^* by moving towards vertices

The Frank-Wolfe algorithm

Consider the simple problem

$$\min \frac{1}{2}\|x\|_2^2$$
$$\text{s.t. } x \in \operatorname{conv}\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$$

and $x^* = \begin{pmatrix}0\\0\end{pmatrix}$



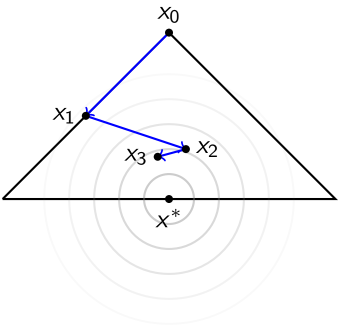
- Let $x_0 = \begin{pmatrix}0\\1\end{pmatrix}$
- FW tries to reach x^* by moving towards vertices

The Frank-Wolfe algorithm

Consider the simple problem

$$\min \frac{1}{2}\|x\|_2^2$$
$$\text{s.t. } x \in \operatorname{conv}\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$$

and $x^* = \begin{pmatrix}0\\0\end{pmatrix}$



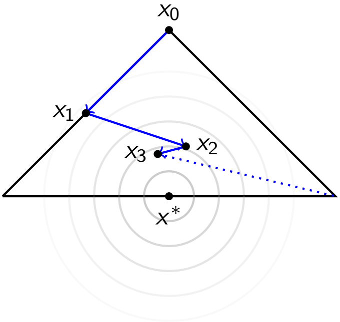
- Let $x_0 = \begin{pmatrix}0\\1\end{pmatrix}$
- FW tries to reach x^* by moving towards vertices

The Frank-Wolfe algorithm

Consider the simple problem

$$\min \frac{1}{2}\|x\|_2^2$$
$$\text{s.t. } x \in \operatorname{conv}\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$$

and $x^* = \begin{pmatrix}0\\0\end{pmatrix}$



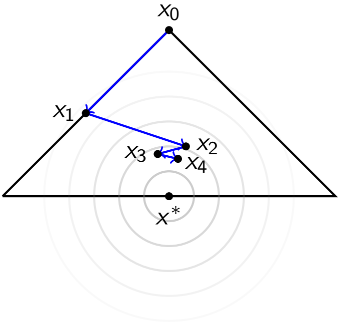
- Let $x_0 = \begin{pmatrix}0\\1\end{pmatrix}$
- FW tries to reach x^* by moving towards vertices

The Frank-Wolfe algorithm

Consider the simple problem

$$\min \frac{1}{2}\|x\|_2^2$$
$$\text{s.t. } x \in \operatorname{conv}\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$$

and $x^* = \begin{pmatrix}0\\0\end{pmatrix}$



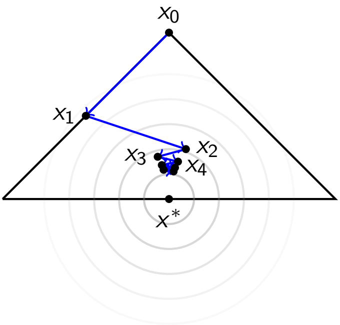
- Let $x_0 = \begin{pmatrix}0\\1\end{pmatrix}$
- FW tries to reach x^* by moving towards vertices

The Frank-Wolfe algorithm

Consider the simple problem

$$\min \frac{1}{2}\|x\|_2^2$$
$$\text{s.t. } x \in \operatorname{conv}\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$$

and $x^* = \begin{pmatrix}0\\0\end{pmatrix}$



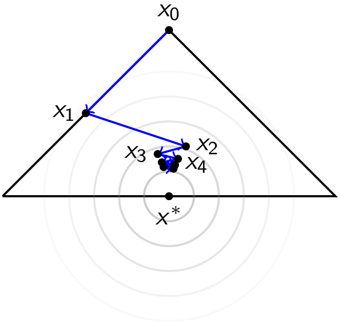
- Let $x_0 = \begin{pmatrix}0\\1\end{pmatrix}$
- FW tries to reach x^* by moving towards vertices

The Frank-Wolfe algorithm

Consider the simple problem

$$\min \frac{1}{2}\|x\|_2^2$$
$$\text{s.t. } x \in \operatorname{conv}\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right)$$

and $x^* = \begin{pmatrix}0\\0\end{pmatrix}$



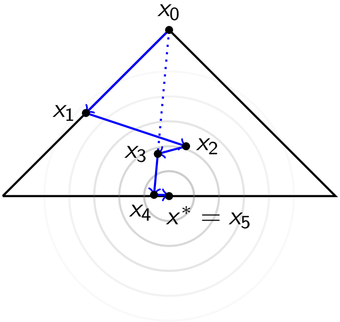
- Let $x_0 = \begin{pmatrix}0\\1\end{pmatrix}$
- FW tries to reach x^* by moving towards vertices
- This yields an inefficient zig-zagging trajectory

Improved Frank-Wolfe variants

- Away-Step Frank-Wolfe (AFW) (Wolfe, 1970; Lacoste-Julien & Jaggi, 2015): enhances FW by allowing to move away from vertices

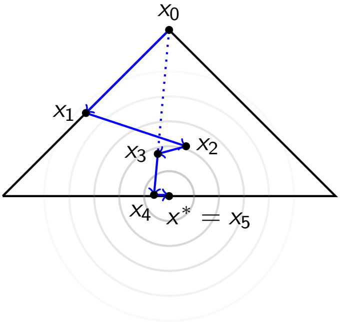
Improved Frank-Wolfe variants

- Away-Step Frank-Wolfe (AFW) (Wolfe, 1970; Lacoste-Julien & Jaggi, 2015): enhances FW by allowing to move away from vertices



Improved Frank-Wolfe variants

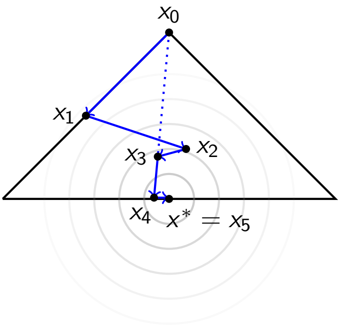
- Away-Step Frank-Wolfe (AFW) (Wolfe, 1970; Lacoste-Julien & Jaggi, 2015): enhances FW by allowing to move away from vertices



- Decomposition-Invariant Pairwise Conditional Gradient (DICG) (Garber & Meshi, 2016): memory-free variant of AFW

Improved Frank-Wolfe variants

- Away-Step Frank-Wolfe (AFW) (Wolfe, 1970; Lacoste-Julien & Jaggi, 2015): enhances FW by allowing to move away from vertices



- Decomposition-Invariant Pairwise Conditional Gradient (DICG) (Garber & Meshi, 2016): memory-free variant of AFW
- Blended Conditional Gradients (BCG) (Braun et al., 2019): blends FCFW and FW

Boosting Frank-Wolfe

- Can we speed up FW in a simple way?

Boosting Frank-Wolfe

- Can we speed up FW in a simple way?
- Rule of thumb in optimization: follow the steepest direction

Boosting Frank-Wolfe

- Can we speed up FW in a simple way?
- Rule of thumb in optimization: follow the steepest direction

Idea:

- Speed up FW by moving in a direction better aligned with $-\nabla f(x_t)$

Boosting Frank-Wolfe

- Can we speed up FW in a simple way?
- Rule of thumb in optimization: follow the steepest direction

Idea:

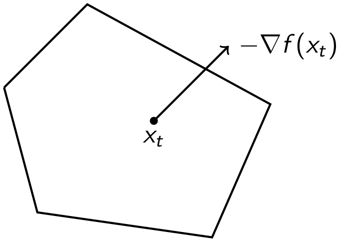
- Speed up FW by moving in a direction better aligned with $-\nabla f(x_t)$
- Build this direction by using $\mathcal{V}$ to maintain the projection-free property

Boosting Frank-Wolfe

- How can we build a direction better aligned with $-\nabla f(x_t)$ and that allows to update x_{t+1} without projection?

Boosting Frank-Wolfe

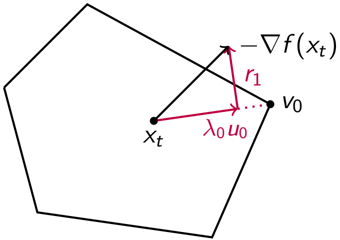
- How can we build a direction better aligned with $-\nabla f(x_t)$ and that allows to update x_{t+1} without projection?



Boosting Frank-Wolfe

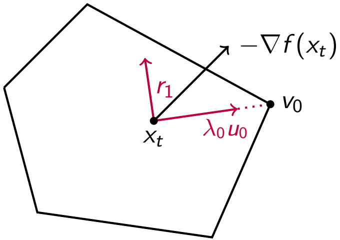
- How can we build a direction better aligned with $-\nabla f(x_t)$ and that allows to update x_{t+1} without projection?

- $v_0 \in \arg\max_{v\in\mathcal{V}} \langle -\nabla f(x_t), v\rangle$
  $\lambda_0 u_0 = \frac{\langle -\nabla f(x_t), v_0 - x_t\rangle}{\|v_0 - x_t\|^2}(v_0 - x_t)$
  $r_1 = -\nabla f(x_t) - \lambda_0 u_0$



Boosting Frank-Wolfe

- How can we build a direction better aligned with $-\nabla f(x_t)$ and that allows to update x_{t+1} without projection?
- $v_0 \in \arg\max_{v \in \mathcal{V}} \langle -\nabla f(x_t), v \rangle$
  $\lambda_0 u_0 = \frac{\langle -\nabla f(x_t), v_0 - x_t \rangle}{\|v_0 - x_t\|^2}(v_0 - x_t)$
  $r_1 = -\nabla f(x_t) - \lambda_0 u_0$

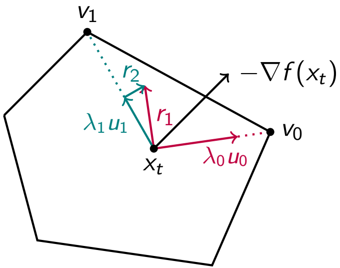


Boosting Frank-Wolfe

- How can we build a direction better aligned with $-\nabla f(x_t)$ and that allows to update x_{t+1} without projection?

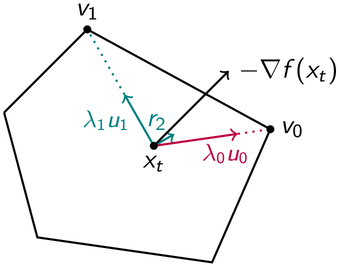
- $v_0 \in \arg\max_{v\in\mathcal{V}} \langle -\nabla f(x_t), v\rangle$
  $\lambda_0 u_0 = \frac{\langle -\nabla f(x_t), v_0 - x_t\rangle}{\|v_0 - x_t\|^2}(v_0 - x_t)$
  $r_1 = -\nabla f(x_t) - \lambda_0 u_0$

- $v_1 \in \arg\max_{v\in\mathcal{V}} \langle r_1, v\rangle$
  $\lambda_1 u_1 = \frac{\langle r_1, v_1 - x_t\rangle}{\|v_1 - x_t\|^2}(v_1 - x_t)$
  $r_2 = r_1 - \lambda_1 u_1$



Boosting Frank-Wolfe

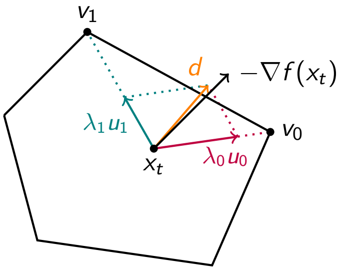
- How can we build a direction better aligned with $-\nabla f(x_t)$ and that allows to update x_{t+1} without projection?
- $v_0 \in \arg\max_{v \in \mathcal{V}} \langle -\nabla f(x_t), v \rangle$
  $\lambda_0 u_0 = \frac{\langle -\nabla f(x_t), v_0 - x_t \rangle}{\|v_0 - x_t\|^2}(v_0 - x_t)$
  $r_1 = -\nabla f(x_t) - \lambda_0 u_0$
- $v_1 \in \arg\max_{v \in \mathcal{V}} \langle r_1, v \rangle$
  $\lambda_1 u_1 = \frac{\langle r_1, v_1 - x_t \rangle}{\|v_1 - x_t\|^2}(v_1 - x_t)$
  $r_2 = r_1 - \lambda_1 u_1$
- We could continue:
  $v_2 \in \arg\max_{v \in \mathcal{V}} \langle r_2, v \rangle$



Boosting Frank-Wolfe

- How can we build a direction better aligned with $-\nabla f(x_t)$ and that allows to update x_{t+1} without projection?

- $v_0 \in \arg\max_{v \in \mathcal{V}} \langle -\nabla f(x_t), v \rangle$
  $\lambda_0 u_0 = \frac{\langle -\nabla f(x_t), v_0 - x_t \rangle}{\|v_0 - x_t\|^2}(v_0 - x_t)$
  $r_1 = -\nabla f(x_t) - \lambda_0 u_0$
- $v_1 \in \arg\max_{v \in \mathcal{V}} \langle r_1, v \rangle$
  $\lambda_1 u_1 = \frac{\langle r_1, v_1 - x_t \rangle}{\|v_1 - x_t\|^2}(v_1 - x_t)$
  $r_2 = r_1 - \lambda_1 u_1$
- We could continue:
  $v_2 \in \arg\max_{v \in \mathcal{V}} \langle r_2, v \rangle$
- $d = \lambda_0 u_0 + \lambda_1 u_1$



Boosting Frank-Wolfe

- How can we build a direction better aligned with $-\nabla f(x_t)$ and that allows to update x_{t+1} without projection?

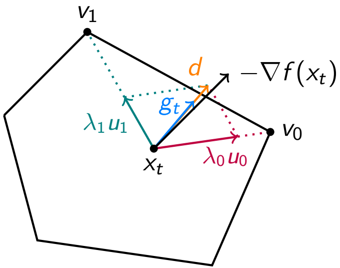
- $v_0 \in \arg\max_{v\in\mathcal{V}} \langle -\nabla f(x_t), v\rangle$
  $\lambda_0 u_0 = \frac{\langle -\nabla f(x_t), v_0 - x_t\rangle}{\|v_0 - x_t\|^2}(v_0 - x_t)$
  $r_1 = -\nabla f(x_t) - \lambda_0 u_0$

- $v_1 \in \arg\max_{v\in\mathcal{V}} \langle r_1, v\rangle$
  $\lambda_1 u_1 = \frac{\langle r_1, v_1 - x_t\rangle}{\|v_1 - x_t\|^2}(v_1 - x_t)$
  $r_2 = r_1 - \lambda_1 u_1$

- We could continue:
  $v_2 \in \arg\max_{v\in\mathcal{V}} \langle r_2, v\rangle$

- $d = \lambda_0 u_0 + \lambda_1 u_1$

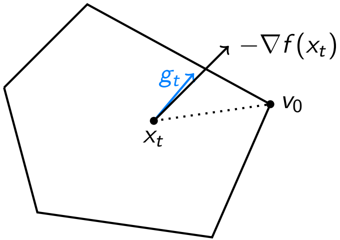
- $g_t = d/(\lambda_0 + \lambda_1)$



Boosting Frank-Wolfe

- How can we build a direction better aligned with $-\nabla f(x_t)$ and that allows to update x_{t+1} without projection?

- $v_0 \in \arg\max_{v \in \mathcal{V}} \langle -\nabla f(x_t), v \rangle$
  $\lambda_0 u_0 = \frac{\langle -\nabla f(x_t), v_0 - x_t \rangle}{\|v_0 - x_t\|^2}(v_0 - x_t)$
  $r_1 = -\nabla f(x_t) - \lambda_0 u_0$
- $v_1 \in \arg\max_{v \in \mathcal{V}} \langle r_1, v \rangle$
  $\lambda_1 u_1 = \frac{\langle r_1, v_1 - x_t \rangle}{\|v_1 - x_t\|^2}(v_1 - x_t)$
  $r_2 = r_1 - \lambda_1 u_1$
- We could continue:
  $v_2 \in \arg\max_{v \in \mathcal{V}} \langle r_2, v \rangle$
- $d = \lambda_0 u_0 + \lambda_1 u_1$
- $g_t = d/(\lambda_0 + \lambda_1)$

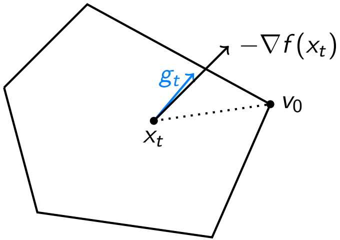


- The boosted direction g_t is better aligned with $-\nabla f(x_t)$ than is the FW direction $v_0 - x_t$

Boosting Frank-Wolfe

- How can we build a direction better aligned with $-\nabla f(x_t)$ and that allows to update x_{t+1} without projection?

- $v_0 \in \arg\max_{v \in \mathcal{V}} \langle -\nabla f(x_t), v \rangle$
  $\lambda_0 u_0 = \frac{\langle -\nabla f(x_t), v_0 - x_t \rangle}{\|v_0 - x_t\|^2}(v_0 - x_t)$
  $r_1 = -\nabla f(x_t) - \lambda_0 u_0$
- $v_1 \in \arg\max_{v \in \mathcal{V}} \langle r_1, v \rangle$
  $\lambda_1 u_1 = \frac{\langle r_1, v_1 - x_t \rangle}{\|v_1 - x_t\|^2}(v_1 - x_t)$
  $r_2 = r_1 - \lambda_1 u_1$
- We could continue:
  $v_2 \in \arg\max_{v \in \mathcal{V}} \langle r_2, v \rangle$
- $d = \lambda_0 u_0 + \lambda_1 u_1$
- $g_t = d/(\lambda_0 + \lambda_1)$



- The boosted direction g_t is better aligned with $-\nabla f(x_t)$ than is the FW direction $v_0 - x_t$ and satisfies $[x_t, x_t + g_t] \subseteq \mathcal{C}$ so we can update

$$x_{t+1} = x_t + \gamma_t g_t \quad \text{for any } \gamma_t \in [0, 1]$$

Boosting Frank-Wolfe

Why $[x_t, x_t + g_t] \subseteq \mathcal{C}$? Let K_t be the number of alignment rounds. We have

$$d = \sum_{k=0}^{K_t - 1} \lambda_k (v_k - x_t) \quad \text{where } \lambda_k > 0 \text{ and } v_k \in \mathcal{V}$$

Boosting Frank-Wolfe

Why $[x_t, x_t + g_t] \subseteq \mathcal{C}$? Let K_t be the number of alignment rounds. We have

$$d = \sum_{k=0}^{K_t-1} \lambda_k(v_k - x_t) \quad \text{where } \lambda_k > 0 \text{ and } v_k \in \mathcal{V}$$

so if $\Lambda_t = \sum_{k=0}^{K-1} \lambda_k$, then

$$g_t = \frac{1}{\Lambda_t} \sum_{k=0}^{K_t-1} \lambda_k(v_k - x_t) = \underbrace{\left(\frac{1}{\Lambda_t} \sum_{k=0}^{K_t-1} \lambda_k v_k\right)}_{\in \mathcal{C}} - x_t$$

Boosting Frank-Wolfe

Why $[x_t, x_t + g_t] \subseteq \mathcal{C}$? Let K_t be the number of alignment rounds. We have

$$d = \sum_{k=0}^{K_t-1} \lambda_k (v_k - x_t) \quad \text{where } \lambda_k > 0 \text{ and } v_k \in \mathcal{V}$$

so if $\Lambda_t = \sum_{k=0}^{K-1} \lambda_k$, then

$$g_t = \frac{1}{\Lambda_t} \sum_{k=0}^{K_t-1} \lambda_k (v_k - x_t) = \underbrace{\left(\frac{1}{\Lambda_t} \sum_{k=0}^{K_t-1} \lambda_k v_k \right)}_{\in \mathcal{C}} - x_t$$

Thus, $x_t + g_t \in \mathcal{C}$ so $[x_t, x_t + g_t] \subseteq \mathcal{C}$ by convexity

Boosting Frank-Wolfe

Algorithm Finding a direction g *well* aligned with ∇ from a reference point z

Input: $z \in \mathcal{C}$, $\nabla \in \mathcal{H}$, $K \in \mathbb{N}\backslash\{0\}$, $\delta \in]0, 1[$.

$d_0 \leftarrow 0$, $\Lambda \leftarrow 0$

for $k = 0$ **to** $K - 1$ **do**

  $r_k \leftarrow \nabla - d_k$ ▷ k-th residual

  $v_k \leftarrow \arg\max_{v \in \mathcal{V}} \langle r_k, v \rangle$ ▷ FW oracle

  $u_k \leftarrow \arg\max_{u \in \{v_k - z, -d_k/\|d_k\|\}} \langle r_k, u \rangle$

  $\lambda_k \leftarrow \langle r_k, u_k \rangle / \|u_k\|^2$

  $d_k' \leftarrow d_k + \lambda_k u_k$

  if $\text{align}(\nabla, d_k') - \text{align}(\nabla, d_k) \geqslant \delta$ **then**

    $d_{k+1} \leftarrow d_k'$

    $$\Lambda_t \leftarrow \begin{cases} \Lambda + \lambda_k & \textbf{if } u_k = v_k - z \\ \Lambda(1 - \lambda_k/\|d_k\|) & \textbf{if } u_k = -d_k/\|d_k\| \end{cases}$$

  else

    break ▷ exit k-loop

$g \leftarrow d_k/\Lambda$ ▷ normalization

Boosting Frank-Wolfe

Algorithm Finding a direction g *well* aligned with ∇ from a reference point z

Input: $z \in \mathcal{C}$, $\nabla \in \mathcal{H}$, $K \in \mathbb{N}\backslash\{0\}$, $\delta \in]0,1[$.

$d_0 \leftarrow 0$, $\Lambda \leftarrow 0$

for $k = 0$ **to** $K-1$ **do**

  $r_k \leftarrow \nabla - d_k$ ▷ k-th residual

  $v_k \leftarrow \arg\max_{v \in \mathcal{V}} \langle r_k, v \rangle$ ▷ FW oracle

  $u_k \leftarrow \arg\max_{u \in \{v_k - z, -d_k/\|d_k\|\}} \langle r_k, u \rangle$

  $\lambda_k \leftarrow \langle r_k, u_k \rangle / \|u_k\|^2$

  $d'_k \leftarrow d_k + \lambda_k u_k$

  if $\operatorname{align}(\nabla, d'_k) - \operatorname{align}(\nabla, d_k) \geqslant \delta$ **then**

    $d_{k+1} \leftarrow d'_k$

    $\Lambda_t \leftarrow \begin{cases} \Lambda + \lambda_k & \textbf{if } u_k = v_k - z \\ \Lambda(1 - \lambda_k/\|d_k\|) & \textbf{if } u_k = -d_k/\|d_k\| \end{cases}$

  else

    break ▷ exit k-loop

$g \leftarrow d_k/\Lambda$ ▷ normalization

- Technicality to ensure convergence of the procedure (Locatello et al., 2017)

Boosting Frank-Wolfe

Algorithm Finding a direction g *well* aligned with ∇ from a reference point z

Input: $z \in \mathcal{C}$, $\nabla \in \mathcal{H}$, $K \in \mathbb{N}\backslash\{0\}$, $\delta \in]0, 1[$.

$d_0 \leftarrow 0$, $\Lambda \leftarrow 0$

for $k = 0$ **to** $K - 1$ **do**

  $r_k \leftarrow \nabla - d_k$ ▷ k-th residual

  $v_k \leftarrow \arg\max_{v \in \mathcal{V}} \langle r_k, v \rangle$ ▷ FW oracle

  $u_k \leftarrow \arg\max_{u \in \{v_k - z, -d_k/\|d_k\|\}} \langle r_k, u \rangle$

  $\lambda_k \leftarrow \langle r_k, u_k \rangle / \|u_k\|^2$

  $d'_k \leftarrow d_k + \lambda_k u_k$

  if $\text{align}(\nabla, d'_k) - \text{align}(\nabla, d_k) \geqslant \delta$ **then**

    $d_{k+1} \leftarrow d'_k$

    $$\Lambda_t \leftarrow \begin{cases} \Lambda + \lambda_k & \textbf{if } u_k = v_k - z \\ \Lambda(1 - \lambda_k/\|d_k\|) & \textbf{if } u_k = -d_k/\|d_k\| \end{cases}$$

  else

    break ▷ exit k-loop

$g \leftarrow d_k/\Lambda$ ▷ normalization

- Technicality to ensure convergence of the procedure (Locatello et al., 2017)
- The stopping criterion is an alignment improvement condition (typically $\delta = 10^{-3}$ and $K = +\infty$)

Boosting Frank-Wolfe

Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0,1]$.

1: **for** $t = 0$ **to** $T-1$ **do**

2: $\quad v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$

Algorithm Boosted Frank-Wolfe (BoostFW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0,1]$, $K \in \mathbb{N} \backslash \{0\}$, $\delta \in]0,1[$.

1: **for** $t = 0$ **to** $T-1$ **do**

2: $\quad g_t \leftarrow \text{procedure}(x_t, -\nabla f(x_t), K, \delta)$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t g_t$

Boosting Frank-Wolfe

Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0,1]$.

1: **for** $t = 0$ **to** $T-1$ **do**

2: $\quad v_t \leftarrow \operatorname*{arg\,min}_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t(v_t - x_t)$

Algorithm Boosted Frank-Wolfe (BoostFW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0,1]$, $K \in \mathbb{N}\backslash\{0\}$, $\delta \in]0,1[$.

1: **for** $t = 0$ **to** $T-1$ **do**

2: $\quad g_t \leftarrow \text{procedure}(x_t, -\nabla f(x_t), K, \delta)$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t g_t$

Boosting Frank-Wolfe

Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0,1]$.

1: **for** $t = 0$ **to** $T-1$ **do**

2: $\quad v_t \leftarrow \operatorname*{arg\,min}_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$

Algorithm Boosted Frank-Wolfe (BoostFW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0,1]$, $K \in \mathbb{N} \backslash \{0\}$, $\delta \in]0,1[$.

1: **for** $t = 0$ **to** $T-1$ **do**

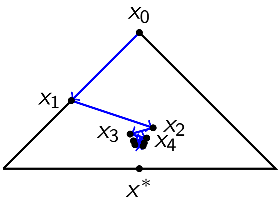
2: $\quad g_t \leftarrow \text{procedure}(x_t, -\nabla f(x_t), K, \delta)$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t g_t$

Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0,1]$.

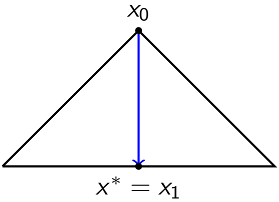
for $t = 0$ **to** $T-1$ **do**
  $v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$
  $x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$



Algorithm Boosted Frank-Wolfe (BoostFW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0,1]$, $K \in \mathbb{N} \setminus \{0\}$, $\delta \in]0,1[$.

for $t = 0$ **to** $T-1$ **do**
  $g_t \leftarrow \text{procedure}(x_t, -\nabla f(x_t), K, \delta)$
  $x_{t+1} \leftarrow x_t + \gamma_t g_t$



Boosting Frank-Wolfe

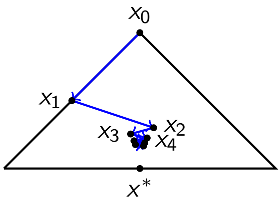
Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0, 1]$.

1: **for** $t = 0$ **to** $T - 1$ **do**

2: $\quad v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$



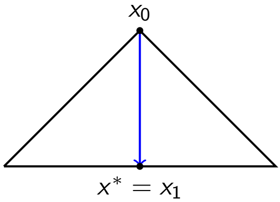
Algorithm Boosted Frank-Wolfe (BoostFW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0, 1]$, $K \in \mathbb{N}\backslash\{0\}$, $\delta \in]0, 1[$.

1: **for** $t = 0$ **to** $T - 1$ **do**

2: $\quad g_t \leftarrow \text{procedure}(x_t, -\nabla f(x_t), K, \delta)$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t g_t$



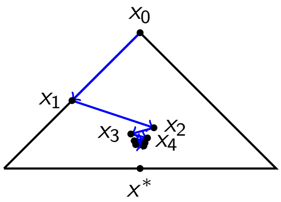
- What is the convergence rate of BoostFW?

Boosting Frank-Wolfe

Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0,1]$.

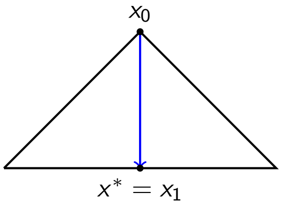
for $t = 0$ **to** $T - 1$ **do**
  $v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$
  $x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$



Algorithm Boosted Frank-Wolfe (BoostFW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0,1]$, $K \in \mathbb{N} \backslash \{0\}$, $\delta \in]0,1[$.

for $t = 0$ **to** $T - 1$ **do**
  $g_t \leftarrow \text{procedure}(x_t, -\nabla f(x_t), K, \delta)$
  $x_{t+1} \leftarrow x_t + \gamma_t g_t$



- What is the convergence rate of BoostFW?
- Is BoostFW expensive in practice?

Boosting Frank-Wolfe

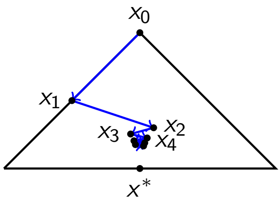
Algorithm Frank-Wolfe (FW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0,1]$.

1: **for** $t = 0$ **to** $T-1$ **do**

2: $\quad v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t (v_t - x_t)$



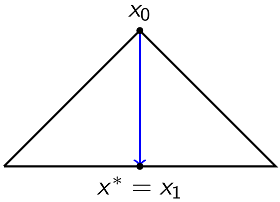
Algorithm Boosted Frank-Wolfe (BoostFW)

Input: $x_0 \in \mathcal{C}$, $\gamma_t \in [0,1]$, $K \in \mathbb{N} \setminus \{0\}$, $\delta \in]0,1[$.

1: **for** $t = 0$ **to** $T-1$ **do**

2: $\quad g_t \leftarrow \text{procedure}(x_t, -\nabla f(x_t), K, \delta)$

3: $\quad x_{t+1} \leftarrow x_t + \gamma_t g_t$



- What is the convergence rate of BoostFW?
- Is BoostFW expensive in practice?
- How does it compare to the state-of-the-art?

Boosting Frank-Wolfe

- Let N_t be the number of iterations up to t where at least 2 rounds of alignment were performed (FW $=$ always 1 round)

Theorem

Let $\mathcal{C} \subset \mathcal{H}$ be a compact convex set with diameter D and $f : \mathcal{H} \to \mathbb{R}$ be a L-smooth, convex, and μ-gradient dominated function, and let $x_0 \in \arg\min_{v \in \mathcal{V}} \langle \nabla f(y), v \rangle$ for some $y \in \mathcal{C}$. Set $\gamma_t = \min \left\{ \frac{\langle -\nabla f(x_t), g_t \rangle}{L \|g_t\|^2}, 1 \right\}$ ("short step") and suppose that $N_t \geqslant \omega t^p$ where $p \in]0, 1]$. Then

$$f(x_t) - \min_{\mathcal{C}} f \leqslant \frac{LD^2}{2} \exp\left(-\delta^2 \frac{\mu}{L} \omega t^p \right)$$

Boosting Frank-Wolfe

- Let N_t be the number of iterations up to t where at least 2 rounds of alignment were performed (FW = always 1 round)

Theorem

Let $\mathcal{C} \subset \mathcal{H}$ be a compact convex set with diameter D and $f : \mathcal{H} \to \mathbb{R}$ be a L-smooth, convex, and μ-gradient dominated function, and let $x_0 \in \arg\min_{v \in \mathcal{V}} \langle \nabla f(y), v \rangle$ for some $y \in \mathcal{C}$. Set $\gamma_t = \min\left\{\frac{\langle -\nabla f(x_t), g_t \rangle}{L\|g_t\|^2}, 1\right\}$ ("short step") and suppose that $N_t \geqslant \omega t^p$ where $p \in \,]0,1]$. Then

$$f(x_t) - \min_{\mathcal{C}} f \leqslant \frac{LD^2}{2} \exp\left(-\delta^2 \frac{\mu}{L} \omega t^p\right)$$

- The assumption $N_t \geqslant \omega t^p$ simply states that N_t is nonnegligeable, i.e., that the boosting procedure is active

Boosting Frank-Wolfe

- Let N_t be the number of iterations up to t where at least 2 rounds of alignment were performed (FW = always 1 round)

Theorem

Let $\mathcal{C} \subset \mathcal{H}$ be a compact convex set with diameter D and $f : \mathcal{H} \to \mathbb{R}$ be a L-smooth, convex, and μ-gradient dominated function, and let $x_0 \in \arg\min_{v \in \mathcal{V}} \langle \nabla f(y), v \rangle$ for some $y \in \mathcal{C}$. Set $\gamma_t = \min\left\{\frac{\langle -\nabla f(x_t), g_t \rangle}{L\|g_t\|^2}, 1\right\}$ ("short step") and suppose that $N_t \geqslant \omega t^p$ where $p \in]0,1]$. Then

$$f(x_t) - \min_{\mathcal{C}} f \leqslant \frac{LD^2}{2} \exp\left(-\delta^2 \frac{\mu}{L} \omega t^p\right)$$

- The assumption $N_t \geqslant \omega t^p$ simply states that N_t is nonnegligeable, i.e., that the boosting procedure is active
- Else BoostFW reduces to FW and the convergence rate is $\frac{4LD^2}{t+2}$

Boosting Frank-Wolfe

- Let N_t be the number of iterations up to t where at least 2 rounds of alignment were performed (FW $=$ always 1 round)

Theorem

Let $\mathcal{C} \subset \mathcal{H}$ be a compact convex set with diameter D and $f : \mathcal{H} \to \mathbb{R}$ be a L-smooth, convex, and μ-gradient dominated function, and let $x_0 \in \arg\min_{v \in \mathcal{V}} \langle \nabla f(y), v \rangle$ for some $y \in \mathcal{C}$. Set $\gamma_t = \min \left\{ \frac{\langle -\nabla f(x_t), g_t \rangle}{L \|g_t\|^2}, 1 \right\}$ ("short step") and suppose that $N_t \geqslant \omega t^p$ where $p \in]0,1]$. Then

$$f(x_t) - \min_{\mathcal{C}} f \leqslant \frac{LD^2}{2} \exp\left(-\delta^2 \frac{\mu}{L} \omega t^p \right)$$

- The assumption $N_t \geqslant \omega t^p$ simply states that N_t is nonnegligeable, i.e., that the boosting procedure is active
- Else BoostFW reduces to FW and the convergence rate is $\frac{4LD^2}{t+2}$
- In practice, $N_t \approx t$ (so $\omega \lesssim 1$ and $p = 1$)

Computational experiments

- We compare BoostFW to AFW, BCG, and DICG on a series of experiments involving various objective functions and feasible regions

Computational experiments

- We compare BoostFW to AFW, BCG, and DICG on a series of experiments involving various objective functions and feasible regions

$$\min_{x \in \mathbb{R}^n} \|y - Ax\|_2^2 \quad \text{s.t.} \quad \|x\|_1 \leqslant \tau$$

$$\begin{aligned}
\min_{x \in \mathbb{R}^{|\mathcal{A}|}} \quad & \sum_{a \in \mathcal{A}} \tau_a x_a \left(1 + 0.03 \left(\frac{x_a}{c_a}\right)^4\right) & \\
\text{s.t.} \quad & x_a = \sum_{r \in \mathcal{R}} \mathbb{1}_{\{a \in r\}} y_r & a \in \mathcal{A} \\
& \sum_{r \in \mathcal{R}_{i,j}} y_r = d_{i,j} & (i,j) \in \mathcal{S} \\
& y_r \geqslant 0 & r \in \mathcal{R}_{i,j},\ (i,j) \in \mathcal{S}
\end{aligned}$$

$$\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^{m} \ln(1 + \exp(-y_i a_i^\top x)) \quad \text{s.t.} \quad \|x\|_1 \leqslant \tau$$

$$\min_{X \in \mathbb{R}^{m \times n}} \frac{1}{|\mathcal{I}|} \sum_{(i,j) \in \mathcal{I}} h_\rho(Y_{i,j} - X_{i,j}) \quad \text{s.t.} \quad \|X\|_{\text{nuc}} \leqslant \tau$$

Computational experiments

- We compare BoostFW to AFW, BCG, and DICG on a series of experiments involving various objective functions and feasible regions

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \; & \|y - Ax\|_2^2 \\ \text{s.t. } & \|x\|_1 \leqslant \tau \end{aligned}$$

$$\begin{aligned} \min_{x\in\mathbb{R}^{|\mathcal{A}|}} \; & \sum_{a\in\mathcal{A}} \tau_a x_a \left(1 + 0.03\left(\frac{x_a}{c_a}\right)^4\right) & \\ \text{s.t. } & x_a = \sum_{r\in\mathcal{R}} \mathbb{1}_{\{a\in r\}} y_r & a \in \mathcal{A} \\ & \sum_{r\in\mathcal{R}_{i,j}} y_r = d_{i,j} & (i,j)\in\mathcal{S} \\ & y_r \geqslant 0 & r\in\mathcal{R}_{i,j},\ (i,j)\in\mathcal{S} \end{aligned}$$

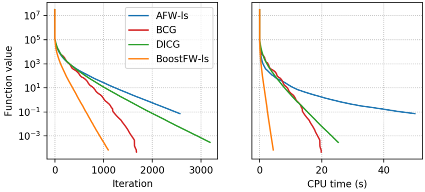
$$\begin{aligned} \min_{x\in\mathbb{R}^n} \; & \frac{1}{m}\sum_{i=1}^{m} \ln(1+\exp(-y_i a_i^\top x)) \\ \text{s.t. } & \|x\|_1 \leqslant \tau \end{aligned}$$

$$\begin{aligned} \min_{X\in\mathbb{R}^{m\times n}} \; & \frac{1}{|\mathcal{I}|}\sum_{(i,j)\in\mathcal{I}} h_\rho(Y_{i,j} - X_{i,j}) \\ \text{s.t. } & \|X\|_{\text{nuc}} \leqslant \tau \end{aligned}$$

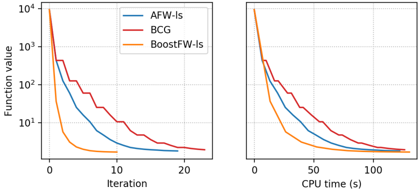
- For BoostFW and AFW we also run the line search-free variants (the "short step" strategy) and label them with an "L"

Computational experiments

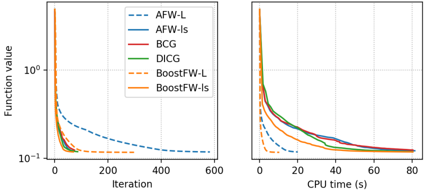
- Sparse signal recovery



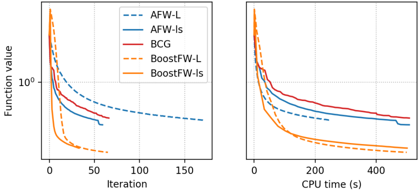
- Traffic assignment



- Sparse logistic regression on the Gisette dataset

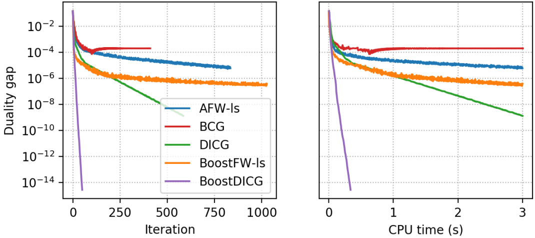


- Collaborative filtering on the MovieLens 100k dataset



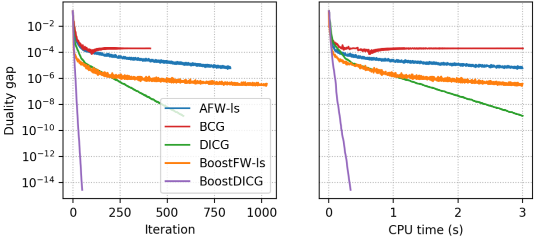
Boosting DICG

- DICG is known to perform particularly well on the video co-localization experiment (YouTube-Objects dataset)
- BoostDICG: application of our method to DICG



Boosting DICG

- DICG is known to perform particularly well on the video co-localization experiment (YouTube-Objects dataset)
- BoostDICG: application of our method to DICG



- (*details*)

DICG

$a_t \leftarrow$ away vertex

$v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$

$x_{t+1} \leftarrow x_t + \gamma_t (v_t - a_t)$

BoostDICG

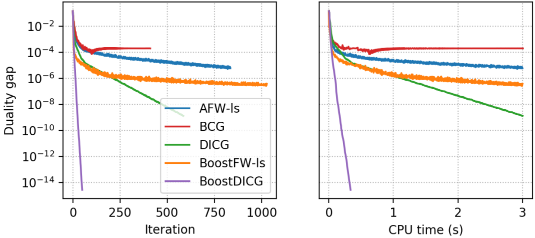
$a_t \leftarrow$ away vertex

$g_t \leftarrow \text{procedure}(a_t, -\nabla f(x_t), K, \delta)$

$x_{t+1} \leftarrow x_t + \gamma_t g_t$

Boosting DICG

- DICG is known to perform particularly well on the video co-localization experiment (YouTube-Objects dataset)
- BoostDICG: application of our method to DICG

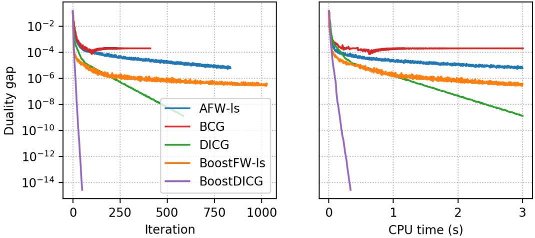


- (*details*)

DICG	BoostDICG
$a_t \leftarrow$ away vertex	$a_t \leftarrow$ away vertex
$v_t \leftarrow \arg\min_{v \in \mathcal{V}} \langle \nabla f(x_t), v \rangle$	$g_t \leftarrow \text{procedure}(a_t, -\nabla f(x_t), K, \delta)$
$x_{t+1} \leftarrow x_t + \gamma_t(v_t - a_t)$	$x_{t+1} \leftarrow x_t + \gamma_t g_t$

Boosting DICG

- DICG is known to perform particularly well on the video co-localization experiment (YouTube-Objects dataset)
- BoostDICG: application of our method to DICG



- (*details*)

DICG	BoostDICG
$a_t \leftarrow$ away vertex	$a_t \leftarrow$ away vertex
$v_t \leftarrow \underset{v \in \mathcal{V}}{\arg\min} \langle \nabla f(x_t), v \rangle$	$g_t \leftarrow \text{procedure}(a_t, -\nabla f(x_t), K, \delta)$
$x_{t+1} \leftarrow x_t + \gamma_t (v_t - a_t)$	$x_{t+1} \leftarrow x_t + \gamma_t g_t$

Takeaways and final remarks

- Projection-free algorithms are of considerable interest in optimization

Takeaways and final remarks

- Projection-free algorithms are of considerable interest in optimization
- We have proposed an intuitive and generic boosting procedure to speed up Frank-Wolfe algorithms

Takeaways and final remarks

- Projection-free algorithms are of considerable interest in optimization
- We have proposed an intuitive and generic boosting procedure to speed up Frank-Wolfe algorithms
- Although our method may perform more linear minimizations per iteration, the progress obtained greatly overcomes their cost

Takeaways and final remarks

- Projection-free algorithms are of considerable interest in optimization
- We have proposed an intuitive and generic boosting procedure to speed up Frank-Wolfe algorithms
- Although our method may perform more linear minimizations per iteration, the progress obtained greatly overcomes their cost
- We focused on smooth convex objective functions, but we expect our method to provide significant gains in performance in other areas of optimization as well

Takeaways and final remarks

- Projection-free algorithms are of considerable interest in optimization
- We have proposed an intuitive and generic boosting procedure to speed up Frank-Wolfe algorithms
- Although our method may perform more linear minimizations per iteration, the progress obtained greatly overcomes their cost
- We focused on smooth convex objective functions, but we expect our method to provide significant gains in performance in other areas of optimization as well

  E.g., large-scale finite-sum/stochastic constrained optimization:

$$
\begin{aligned}
&g_t \leftarrow \text{procedure}(x_t, -\tilde{\nabla} f(x_t), K, \delta) \\
&x_{t+1} \leftarrow x_t + \gamma_t g_t
\end{aligned}
$$

References

G. Braun, S. Pokutta, D. Tu, and S. Wright. Blended conditional gradients: the unconditioning of conditional gradients. *ICML*, 2019.

M. D. Canon and C. D. Cullum. A tight upper bound on the rate of convergence of Frank-Wolfe algorithm. *SIAM J. Control*, 1968.

C. W. Combettes and S. Pokutta. Boosting Frank-Wolfe by chasing gradients. *ICML*, 2020.

M. Frank and P. Wolfe. An algorithm for quadratic programming. *Naval Res. Logist. Q.*, 1956.

D. Garber and O. Meshi. Linear-memory and decomposition-invariant linearly convergent conditional gradient algorithm for structured polytopes. *NIPS*, 2016.

M. Jaggi. Revisiting Frank-Wolfe: Projection-free sparse convex optimization. *ICML*, 2013.

S. Lacoste-Julien and M. Jaggi. On the global linear convergence of Frank-Wolfe optimization variants. *NIPS*, 2015.

G. Lan. The complexity of large-scale convex programming under a linear optimization oracle. Technical report, University of Florida, 2013.

E. S. Levitin and B. T. Polyak. Constrained minimization methods. *USSR Comp. Math. Math. Phys.*, 1966.

F. Locatello, M. Tschannen, G. Rätsch, and M. Jaggi. Greedy algorithms for cone constrained optimization with convergence guarantees. *NIPS*, 2017.

P. Wolfe. Convergence theory in nonlinear programming. *Integer and Nonlinear Programming*. North-Holland, 1970.