# Frank-Wolfe Methods for Optimization and Machine Learning

Cyrille W. Combettes

School of Industrial and Systems Engineering
Georgia Institute of Technology

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# Outline

# Introduction

We consider

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } x \in \mathcal{C} \end{aligned}$$

where

- $\mathcal{C} \subset \mathbb{R}^n$ is a compact convex set
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## Example

- Sparse logistic regression

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^{m} \ln(1 + \exp(-y_i \langle a_i, x \rangle)) \\ &\text{s.t. } \|x\|_1 \leqslant \tau \end{aligned}$$

- Low-rank matrix completion

$$\begin{aligned} &\min_{X \in \mathbb{R}^{m \times n}} \frac{1}{2|\mathcal{I}|} \sum_{(i,j) \in \mathcal{I}} (Y_{i,j} - X_{i,j})^2 \\ &\text{s.t. } \|X\|_{\text{nuc}} \leqslant \tau \end{aligned}$$

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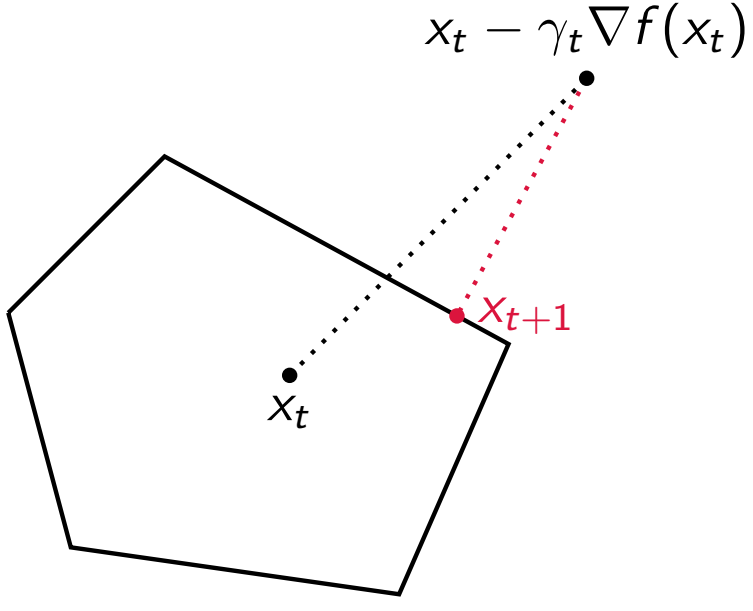

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- This is an issue with the method of projections, not necessarily with the geometry of $\mathcal{C}$: linear minimizations over $\mathcal{C}$ can still be relatively cheap
- We compare

$$\arg\min_{x \in \mathcal{C}} \langle x, y \rangle \quad \text{and} \quad \arg\min_{x \in \mathcal{C}} \|x - y\|$$

on several sets commonly used in optimization

# Complexity of linear minimization and projection

| **Set $\mathcal{C}$** | **Linear minimization** | **Projection** |
|---|---|---|
| $\ell_1/\ell_2/\ell_\infty$-ball | $\mathcal{O}(n)$ | $\mathcal{O}(n)$ |
| $\ell_p$-ball, $p \in ]1, \infty[ \setminus \{2\}$ | $\mathcal{O}(n)$ | $\mathcal{O}(n\rho^2 \|y - x^*\|_2^2/\varepsilon^2)$ |
| Nuclear norm-ball | $\mathcal{O}(\nu \ln(m+n)\sqrt{\sigma_1}/\sqrt{\varepsilon})$ | $\mathcal{O}(mn \min\{m, n\})$ |
| Flow polytope | $\mathcal{O}(m+n)$ | $\tilde{\mathcal{O}}(m^3 n + n^2)$ |
| Birkhoff polytope | $\mathcal{O}(n^3)$ | $\mathcal{O}(n^2 d_z^2/\varepsilon^2)$ |
| Permutahedron | $\mathcal{O}(n \ln(n))$ | $\mathcal{O}(n \ln(n) + n)$ |

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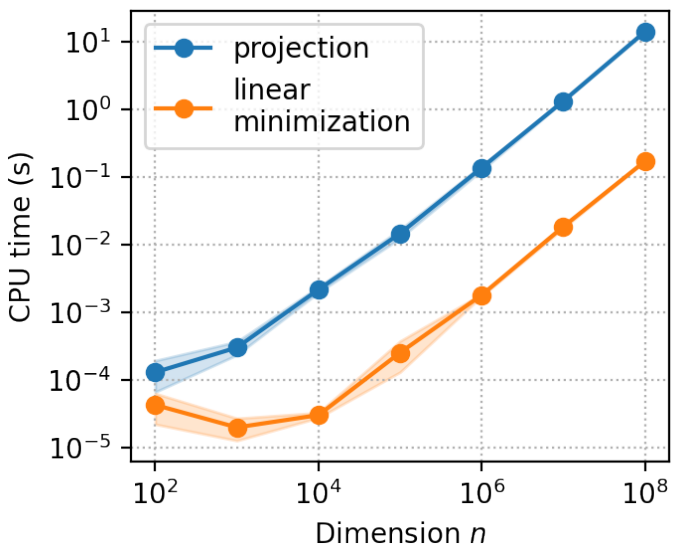
Example: the $\ell_1$-ball



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- Can we avoid projections?

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The Frank-Wolfe algorithm (Frank & Wolfe, 1956) a.k.a. conditional gradient algorithm (Levitin & Polyak, 1966):

**Algorithm** Frank-Wolfe (FW)

**Input:** $x_0 \in \mathcal{C}$, $\gamma_t \in [0, 1]$

1: **for** $t = 0$ **to** $T - 1$ **do**

2: $\quad v_t \leftarrow \arg\min_{v \in \mathcal{C}} \langle v, \nabla f(x_t) \rangle$

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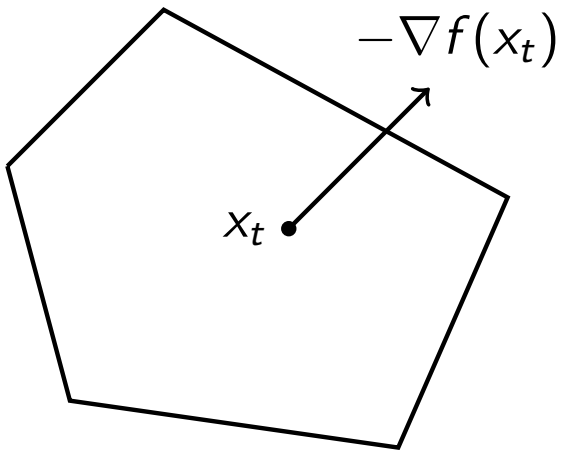

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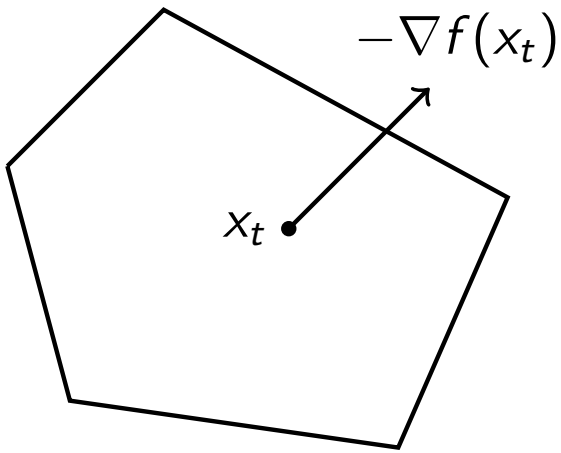

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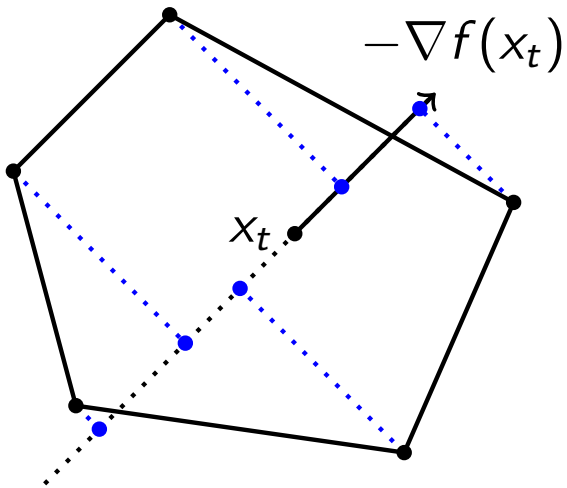

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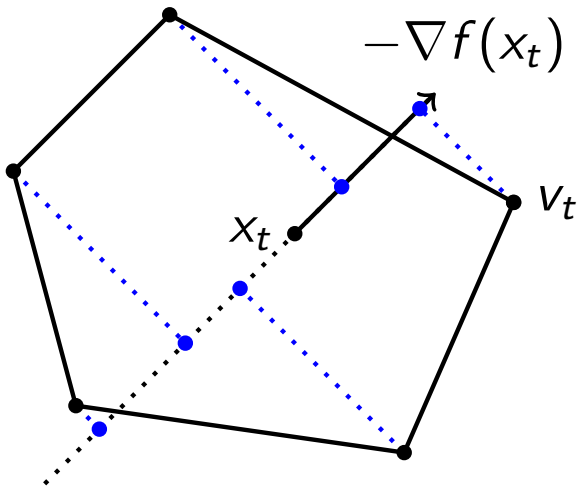

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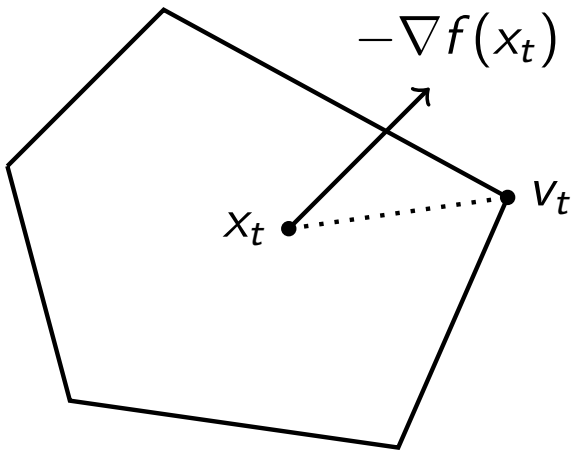

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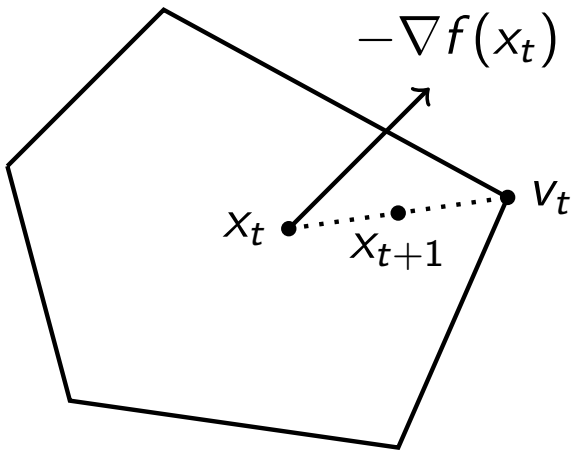

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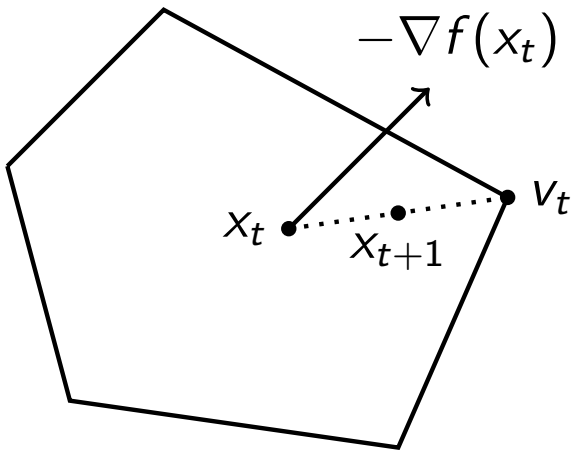


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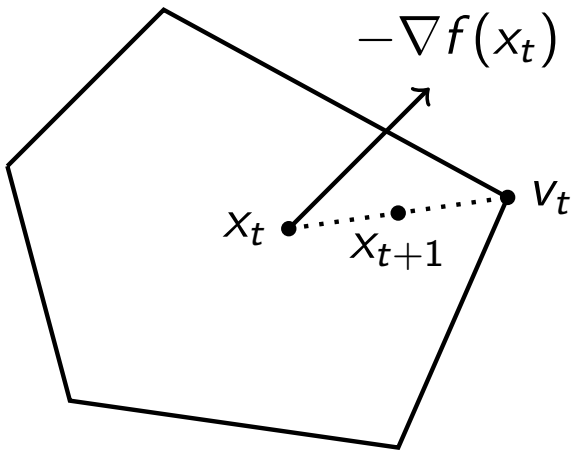


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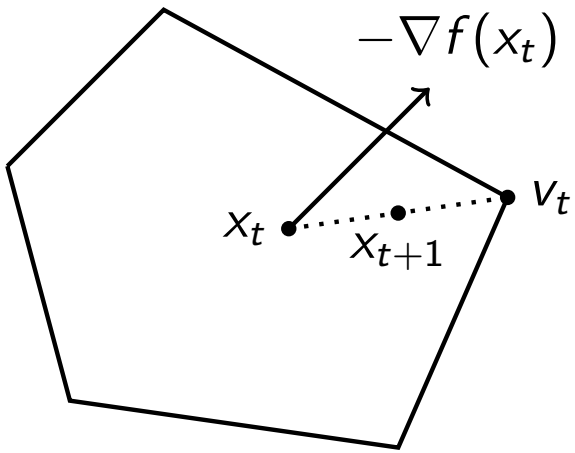


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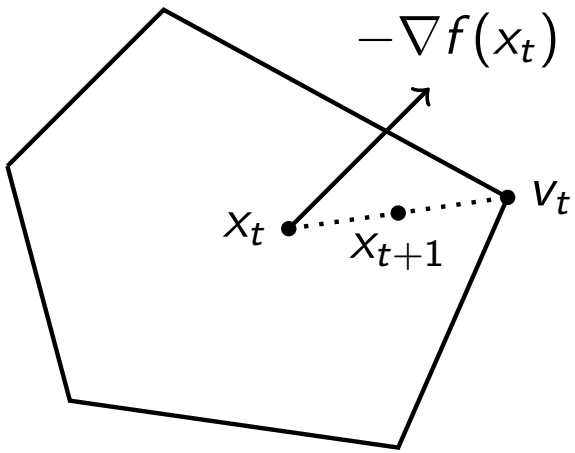


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- Applications: traffic assignment, computer vision, optimal transport, adversarial learning, etc.

# Step-size strategies

- The first strategy considered historically (Frank & Wolfe 1956; Levitin & Polyak, 1966; Demyanov & Rubinov, 1970) is

$$\gamma_t \leftarrow \min\left\{\frac{\langle x_t - v_t, \nabla f(x_t)\rangle}{L\|x_t - v_t\|^2}, 1\right\}$$

and is obtained from the smoothness upper bound:

$$\gamma_t = \underset{\gamma\in[0,1]}{\arg\min}\, f(x_t) + \gamma\langle v_t - x_t, \nabla f(x_t)\rangle + \frac{L}{2}\gamma^2\|v_t - x_t\|_2^2$$

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- Later on, Dunn & Harshbarger (1978) proposed *open-loop* strategies in the form $\gamma_t \sim 1/t$. The one popularized by Jaggi (2013) is

$$\gamma_t \leftarrow \frac{2}{t+2}$$

# Convergence analysis

**Theorem (Frank & Wolfe, 1956; Levitin & Polyak, 1966; Jaggi, 2013)**

*Let $\mathcal{C} \subset \mathbb{R}^n$ be a compact convex set with diameter $D$ and $f : \mathbb{R}^n \to \mathbb{R}$ be an $L$-smooth convex function. Then*

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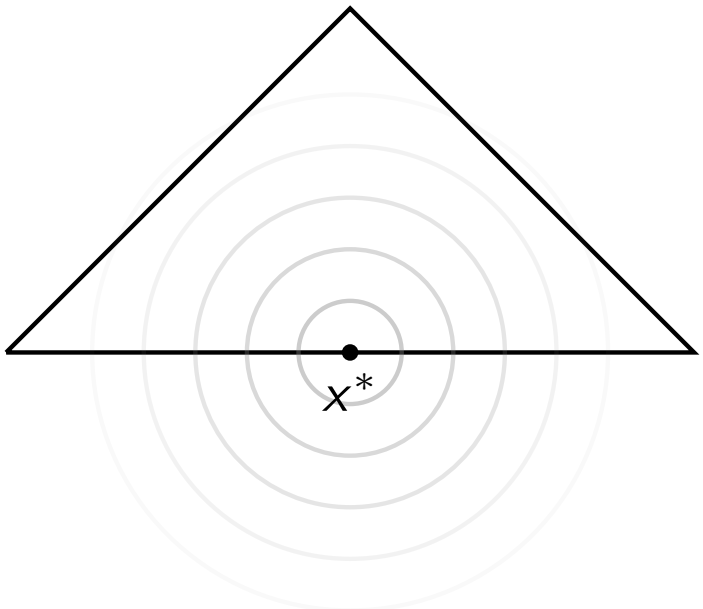
- The convergence rate cannot be improved (Canon & Cullum, 1968; Jaggi, 2013; Lan, 2013)
- The iterates can follow a naive and slow trajectory

# The zigzagging phenomenon

Consider the simple problem

$$\begin{aligned} &\min \frac{1}{2}\|x\|_2^2 \\ &\text{s.t. } x \in \operatorname{conv}\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right\} \end{aligned}$$

and $x^* = \begin{pmatrix}0\\0\end{pmatrix}$



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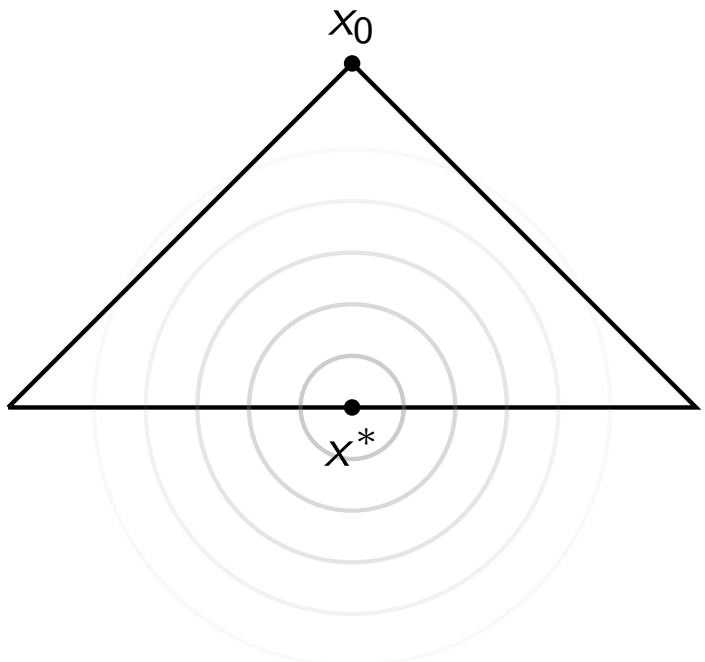


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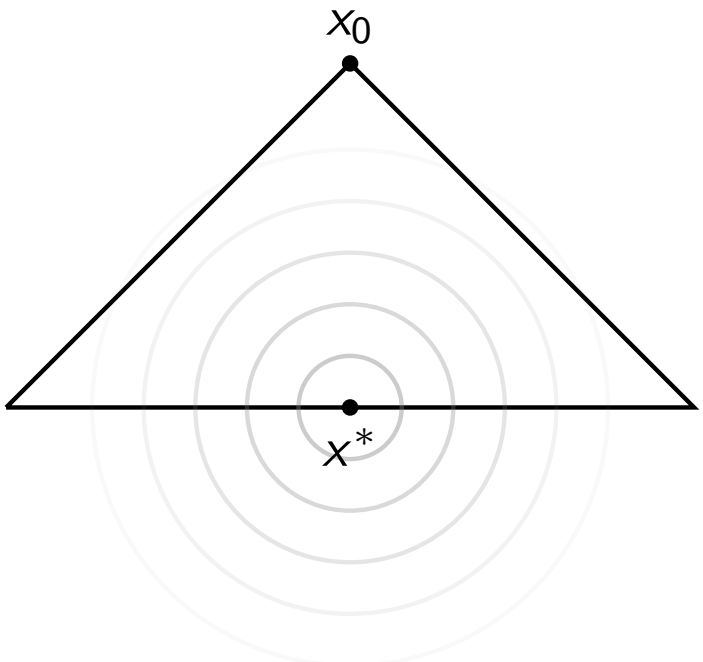


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- FW tries to reach $x^*$ by moving towards vertices

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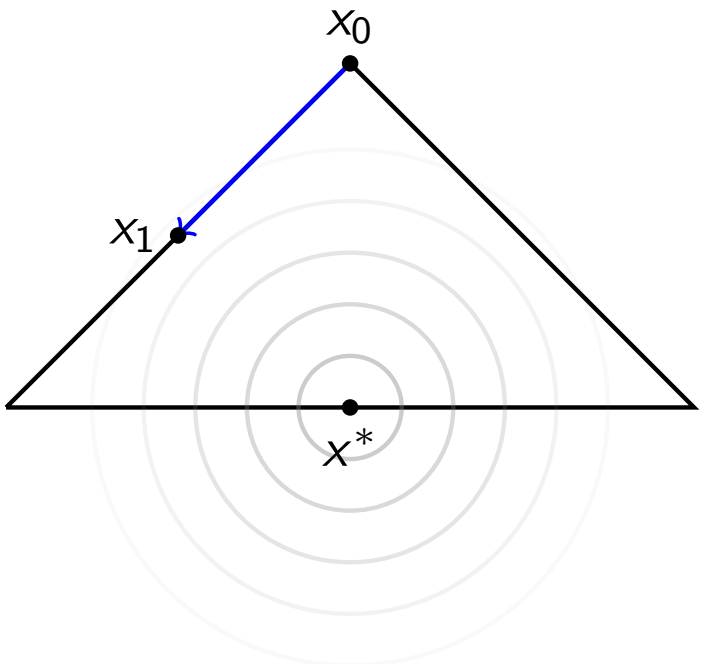


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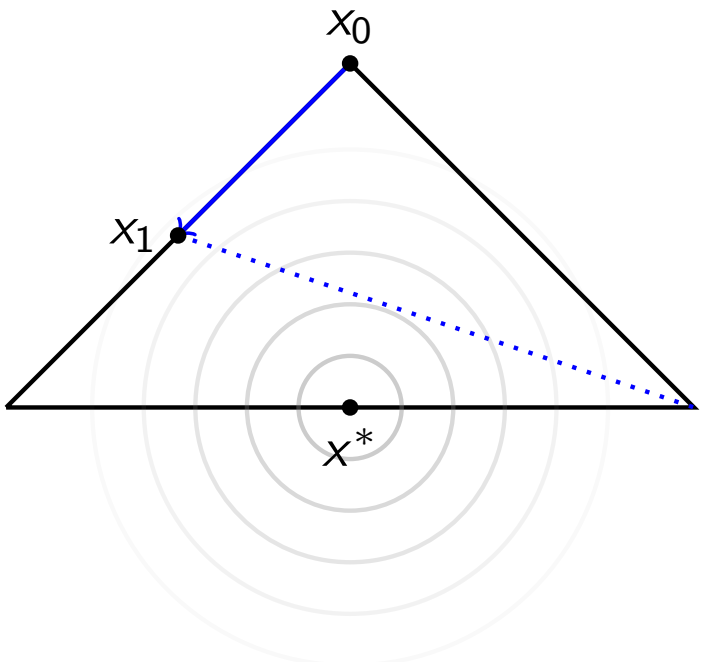


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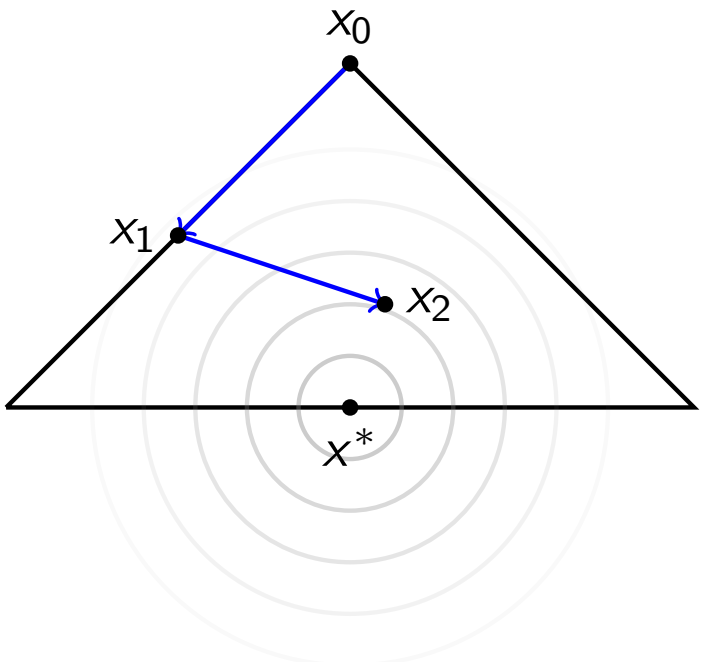


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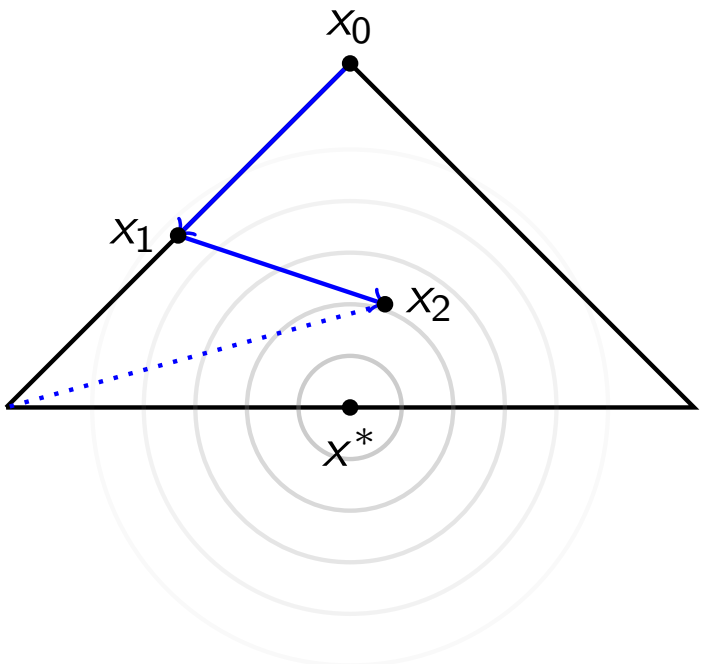


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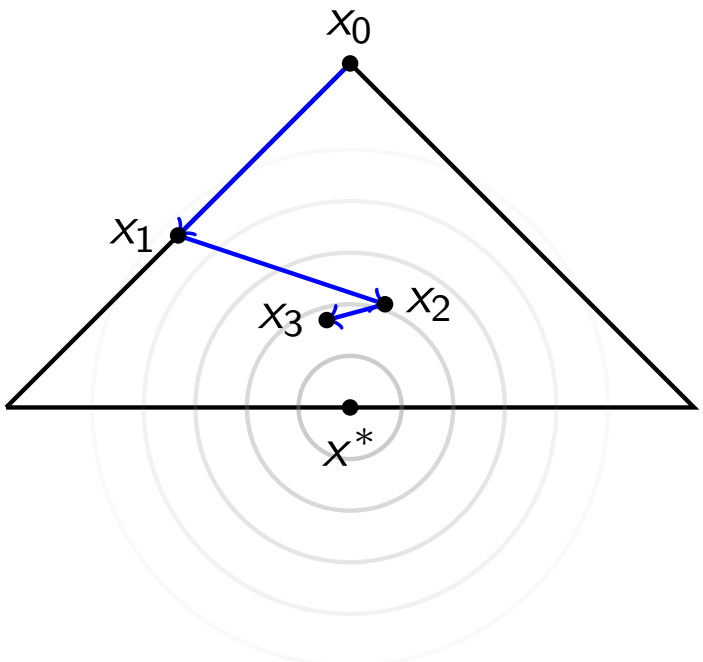


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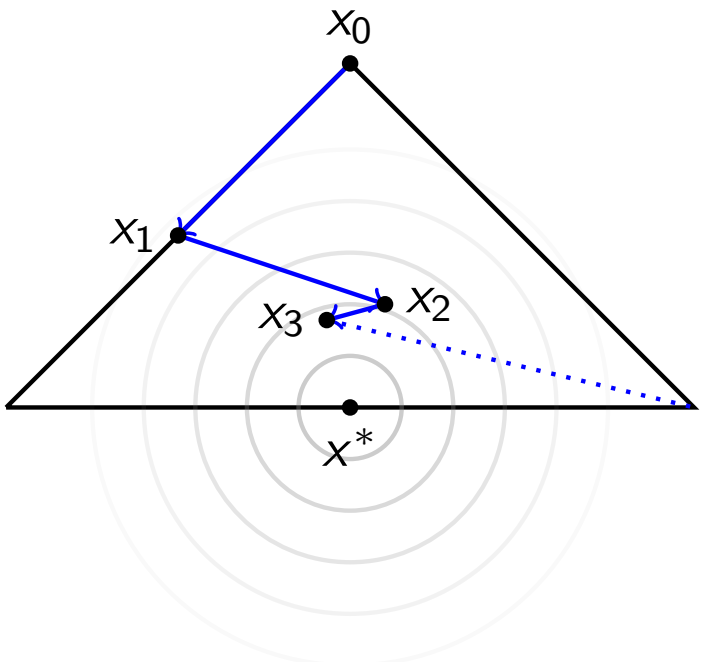


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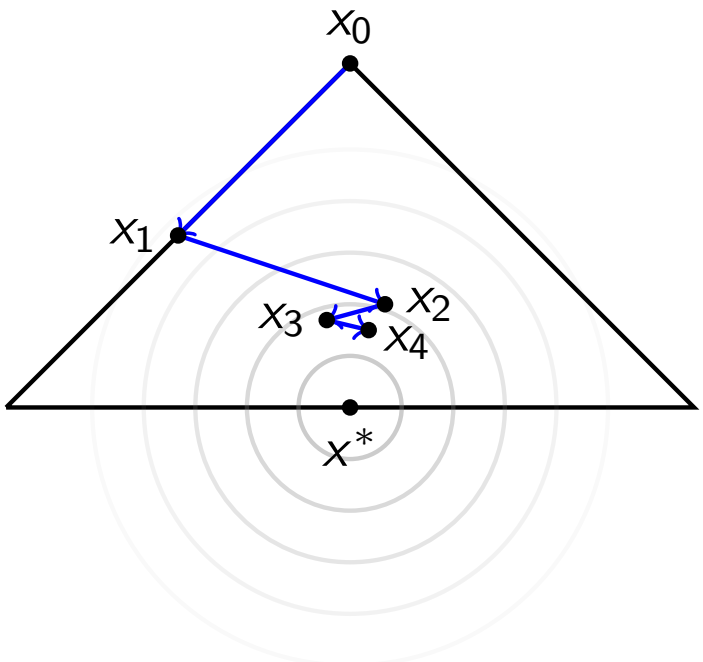


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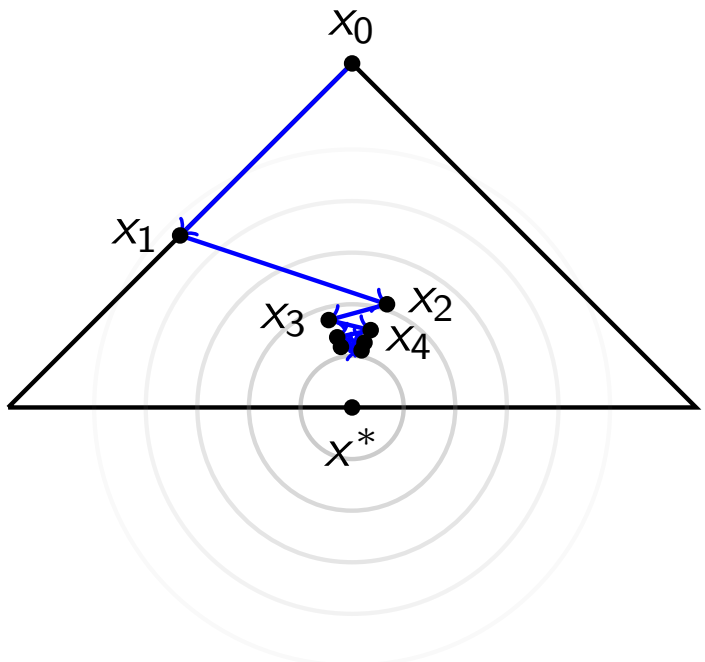


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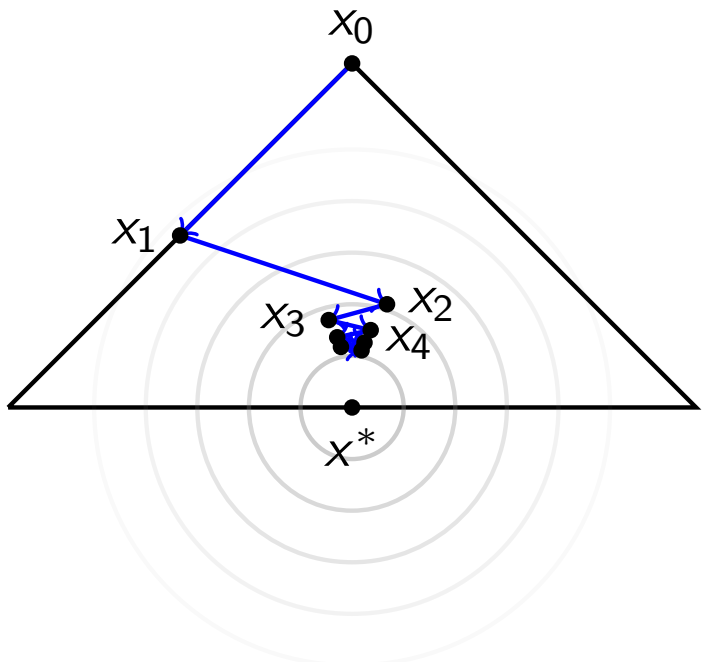


- Let $x_0 = \begin{pmatrix}0\\1\end{pmatrix}$
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- This yields an inefficient zig-zagging trajectory

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- Away-Step Frank-Wolfe (AFW) (Wolfe, 1970; Guélat & Marcotte, 1986; Lacoste-Julien & Jaggi, 2015): enhances FW by allowing to also move away from vertices

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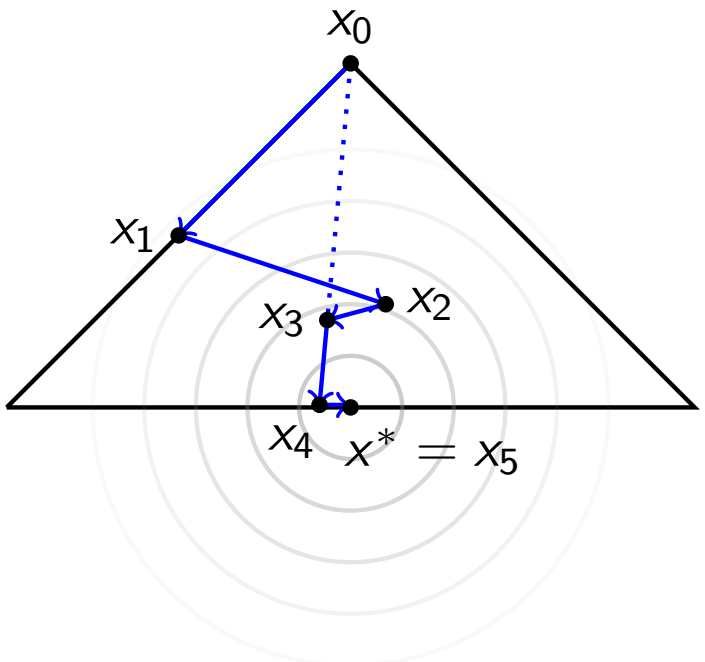

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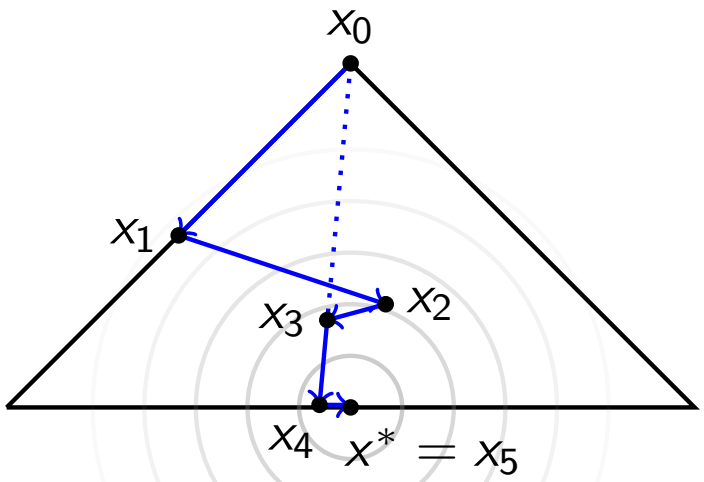


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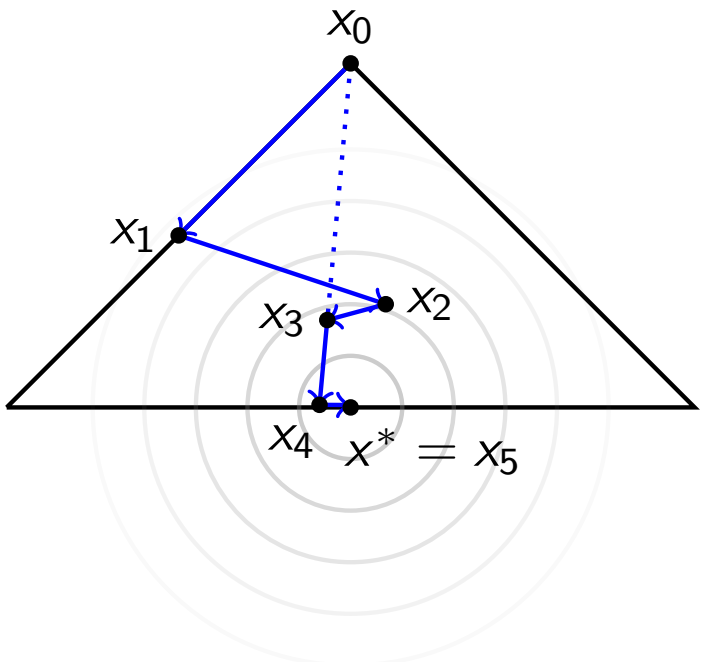


- Decomposition-Invariant Pairwise Conditional Gradient (DICG) (Garber & Meshi, 2016): memory-free variant of AFW
- Blended Conditional Gradients (BCG) (Braun et al., 2019): blends FCFW and FW

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- Build this direction by using $\mathcal{C}$ to maintain the projection-free property

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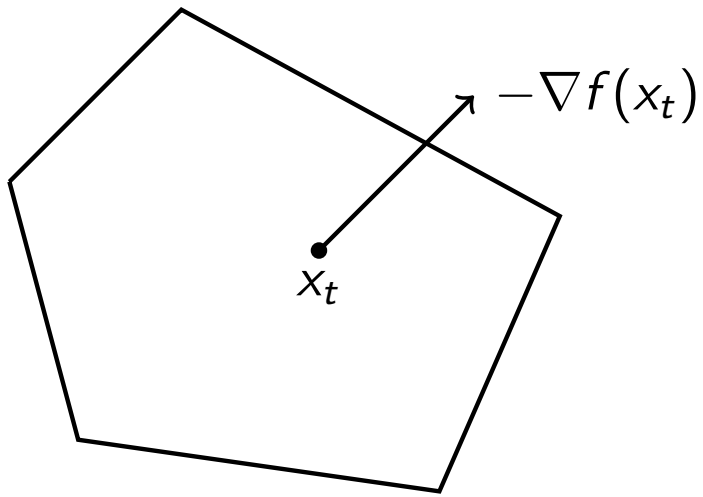

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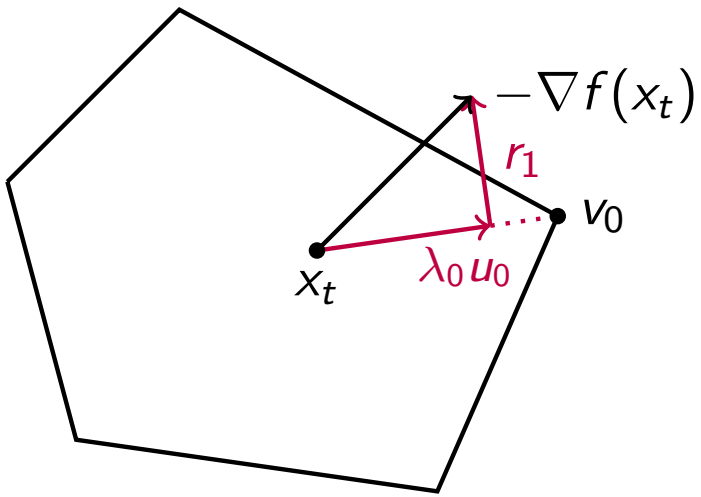

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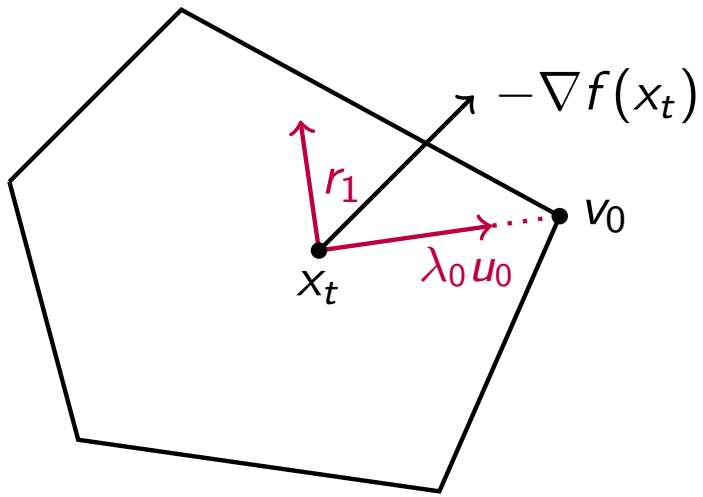

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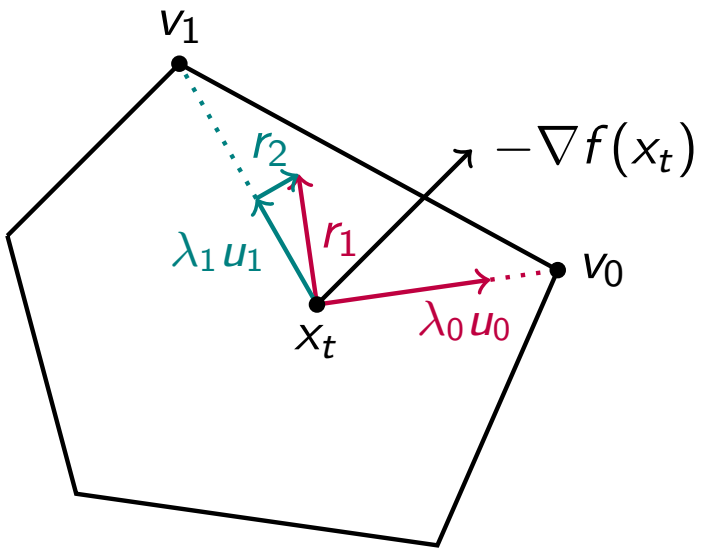

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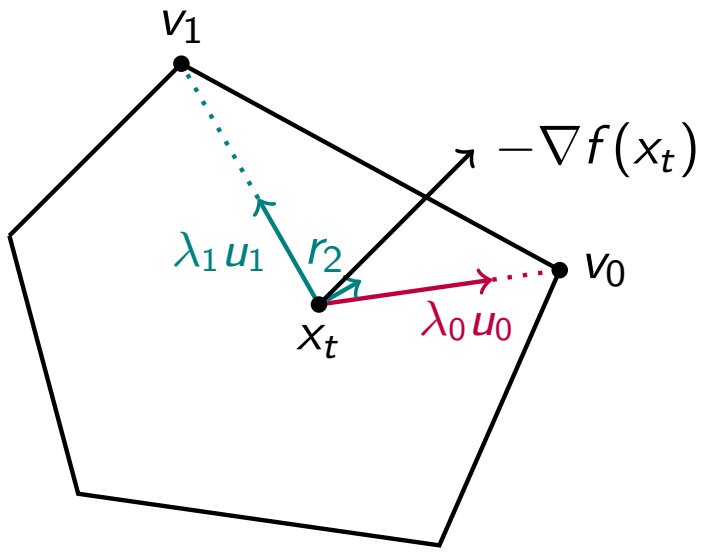

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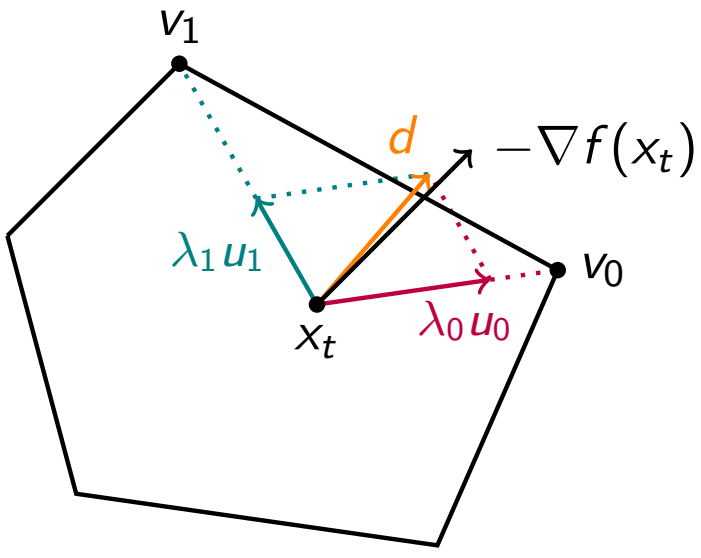

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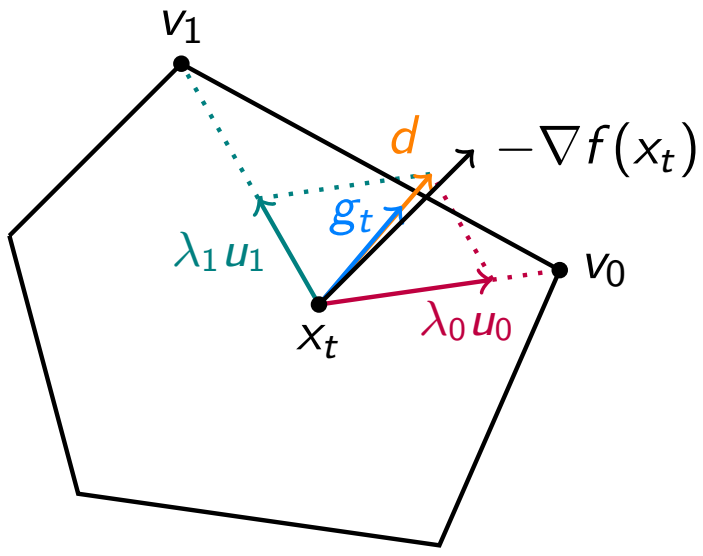

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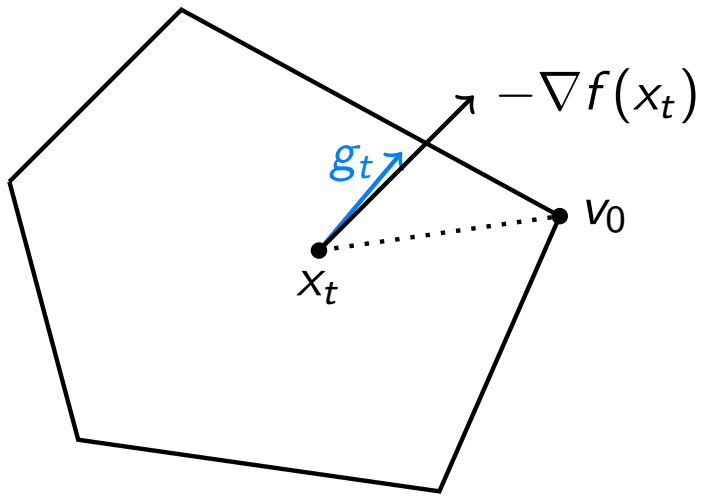


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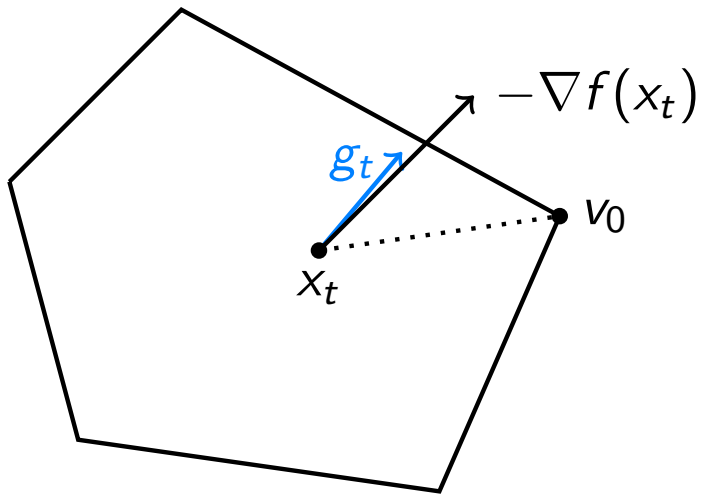


- The boosted direction $g_t$ is better aligned with $-\nabla f(x_t)$ than is the FW direction $v_0 - x_t$ and satisfies $[x_t, x_t + g_t] \subset \mathcal{C}$ so we can update

$$x_{t+1} = x_t + \gamma_t g_t \quad \text{for all } \gamma_t \in [0, 1]$$

# A generic boosting procedure

**Algorithm** Boosting procedure Boost($\mathbf{d}, \mathbf{z}, K, \delta$)

**Input:** $\mathbf{d} \neq 0$, $\mathbf{z} \in \mathcal{C}$, $K \in \mathbb{N}\backslash\{0\}$, $\delta \in ]0,1[$

$d_0 \leftarrow 0$, $\Lambda \leftarrow 0$
**for** $k = 0$ **to** $K - 1$ **do**
  $r_k \leftarrow \mathbf{d} - d_k$ ▷ $k$-th residual
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  $u_k \leftarrow v_k - \mathbf{z}$
  $\lambda_k \leftarrow \langle u_k, r_k \rangle / \|u_k\|_2^2$
  $d'_k \leftarrow d_k + \lambda_k u_k$
  **if** $\cos(d'_k, \mathbf{d}) - \cos(d_k, \mathbf{d}) \geqslant \delta$ **then**
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- The stopping criterion is an alignment improvement condition (typically $\delta \leftarrow 10^{-3}$ and $K \leftarrow +\infty$)

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  **if** $\cos(d'_k, \mathbf{d}) - \cos(d_k, \mathbf{d}) \geqslant \delta$ **then**
    $d_{k+1} \leftarrow d'_k$
    $\Lambda \leftarrow \Lambda + \lambda_k$
  **else**
    **break** ▷ exit $k$-loop
$g \leftarrow d_k / \Lambda$ ▷ normalization

- $\cos(\hat{d}, \mathbf{d}) = \frac{\langle \hat{d}, \mathbf{d} \rangle}{\|\hat{d}\|_2 \|\mathbf{d}\|_2}$ if $\hat{d} \neq 0$, else $-1$ if $\hat{d} = 0$
- The stopping criterion is an alignment improvement condition (typically $\delta \leftarrow 10^{-3}$ and $K \leftarrow +\infty$)

# The Boosted Frank-Wolfe algorithm

**Algorithm** Frank-Wolfe (FW)

**Input:** $x_0 \in \mathcal{C}$, $\gamma_t \in [0, 1]$

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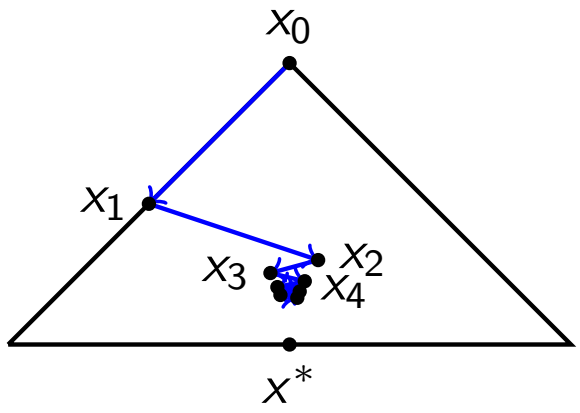


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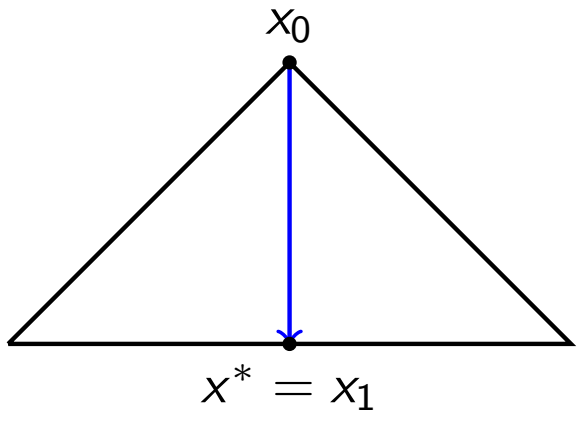

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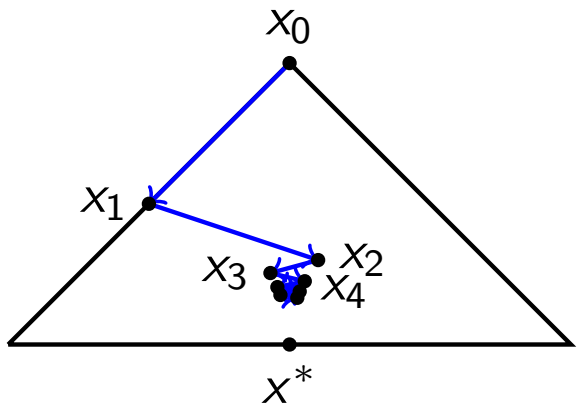


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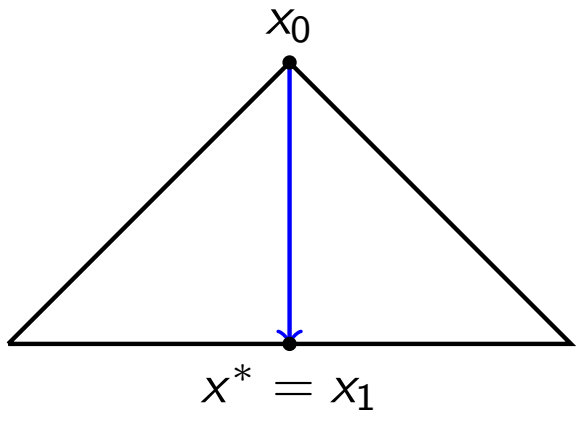


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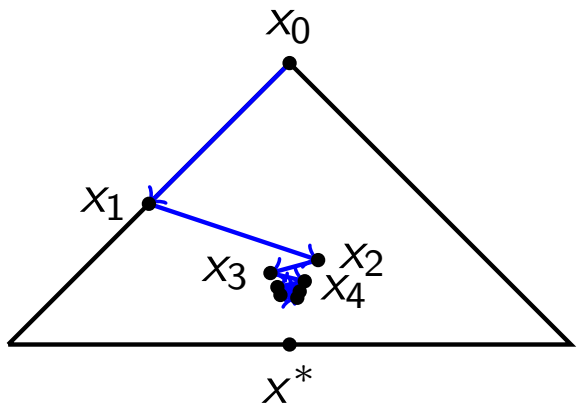


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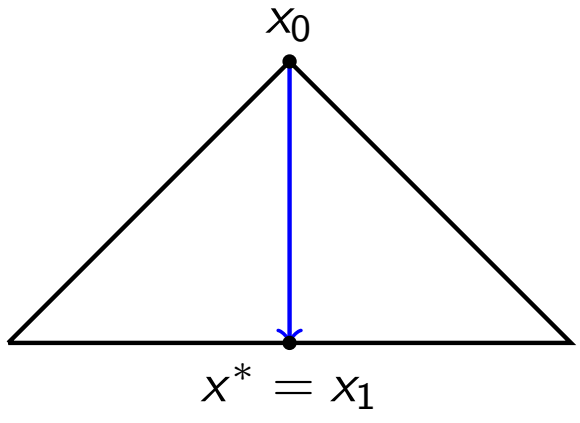


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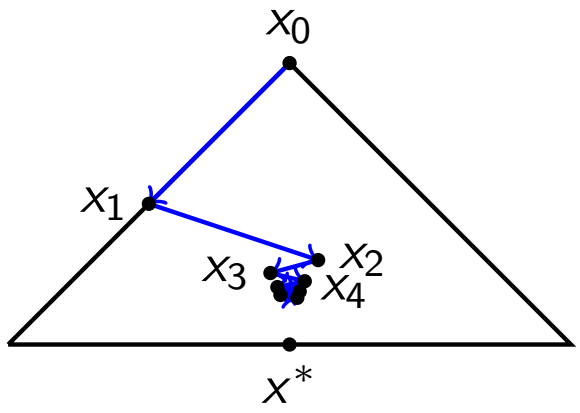


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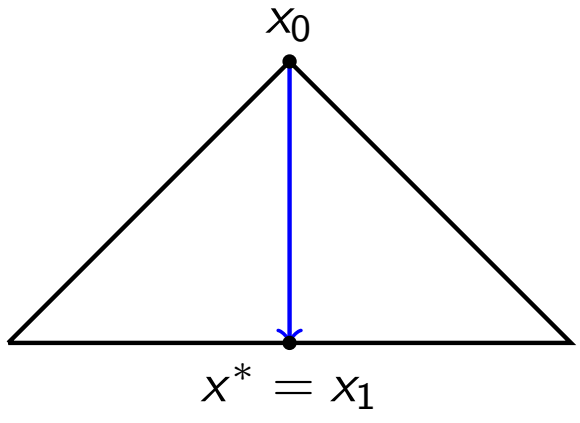


- What is the convergence rate of BoostFW?
- Is BoostFW expensive in practice?
- How does it compare to the state of the art?

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- Let $N_t$ be the number of iterations up to $t$ for which at least 2 rounds of alignment were performed (FW $=$ always 1 round) with a step-size $< 1$

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- Else, BoostFW reduces to FW and the convergence rate is $\frac{4LD^2}{t+2}$
- In practice, $N_t \approx t$ (so $\omega \lesssim 1$)

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$$\min_{x \in \mathbb{R}^{|\mathcal{A}|}} \sum_{a \in \mathcal{A}} \tau_a x_a \left(1 + 0.03 \left(\frac{x_a}{c_a}\right)^4\right)$$
$$\text{s.t. } x_a = \sum_{r \in \mathcal{R}} \mathbb{1}_{\{a \in r\}} y_r \qquad a \in \mathcal{A}$$
$$\sum_{r \in \mathcal{R}_{i,j}} y_r = d_{i,j} \qquad (i,j) \in \mathcal{S}$$
$$y_r \geqslant 0 \qquad r \in \mathcal{R}_{i,j},\ (i,j) \in \mathcal{S}$$

$$\min_{x \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^{m} \ln(1 + \exp(-y_i \langle a_i, x \rangle)) \quad \text{s.t. } \|x\|_1 \leqslant \tau$$

$$\min_{X \in \mathbb{R}^{m \times n}} \frac{1}{|\mathcal{I}|} \sum_{(i,j) \in \mathcal{I}} h_\rho(Y_{i,j} - X_{i,j}) \quad \text{s.t. } \|X\|_{\text{nuc}} \leqslant \tau$$

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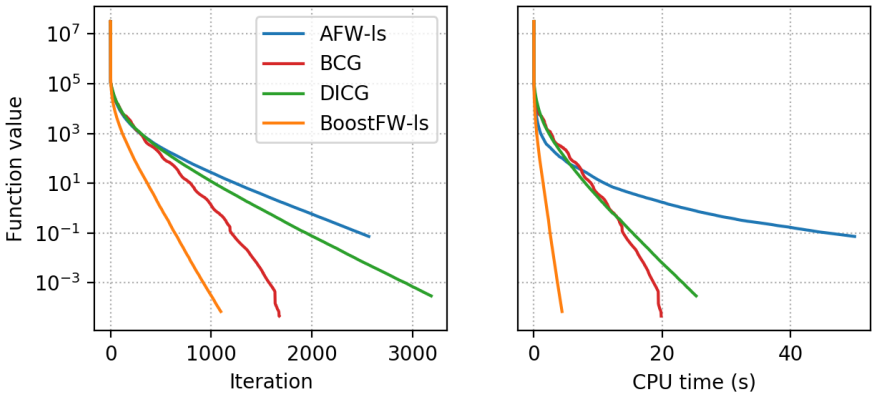
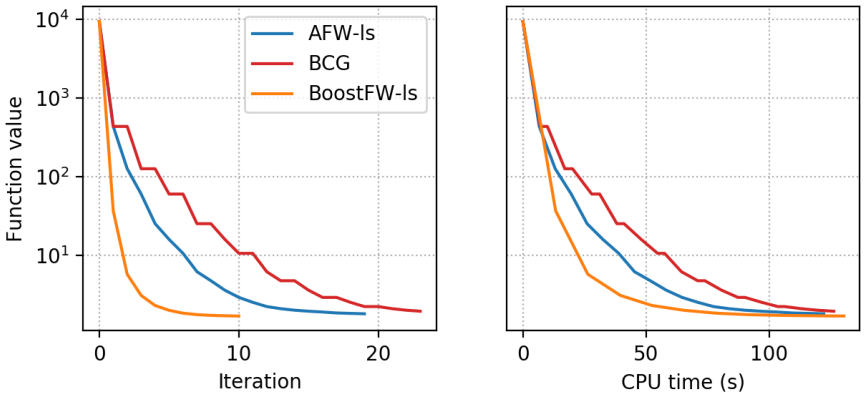
- For BoostFW and AFW we also run the line search-free strategies and label them with an "L"

# Computational experiments

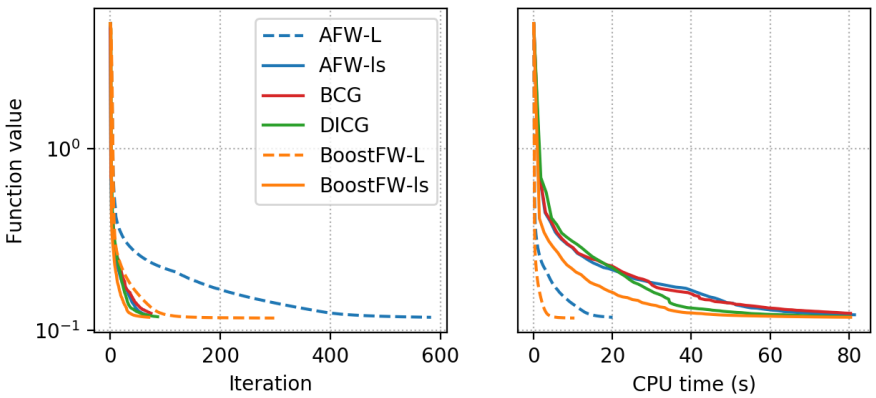
- Sparse signal recovery



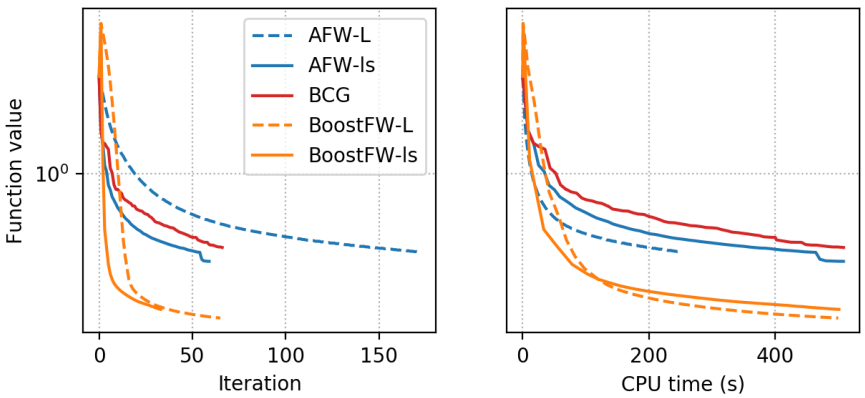

- Traffic assignment




- Sparse logistic regression on the Gisette dataset




- Collaborative filtering on the MovieLens 100k dataset



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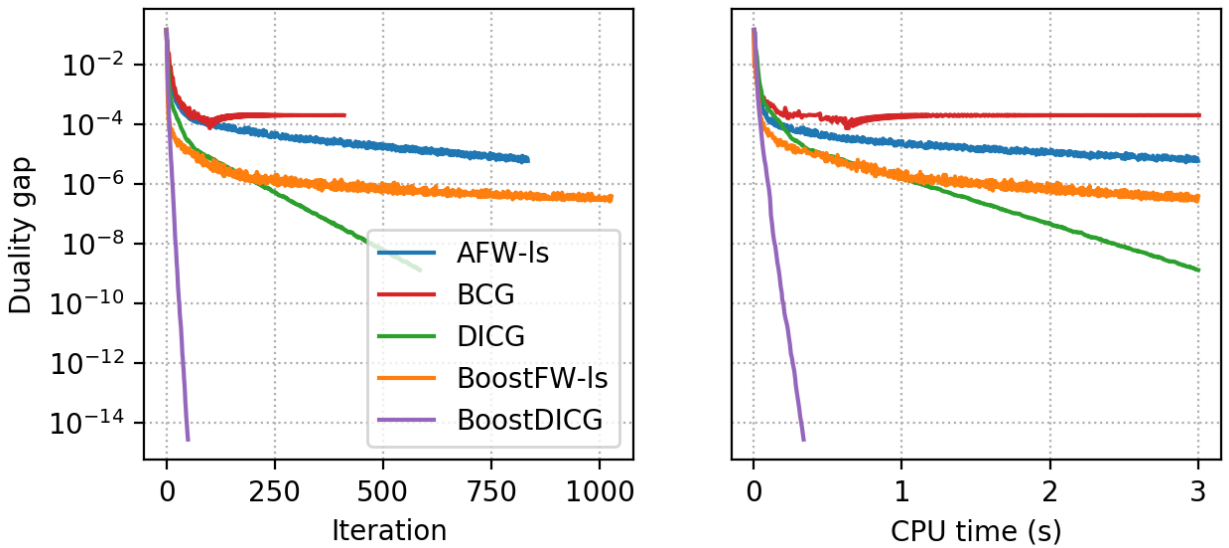

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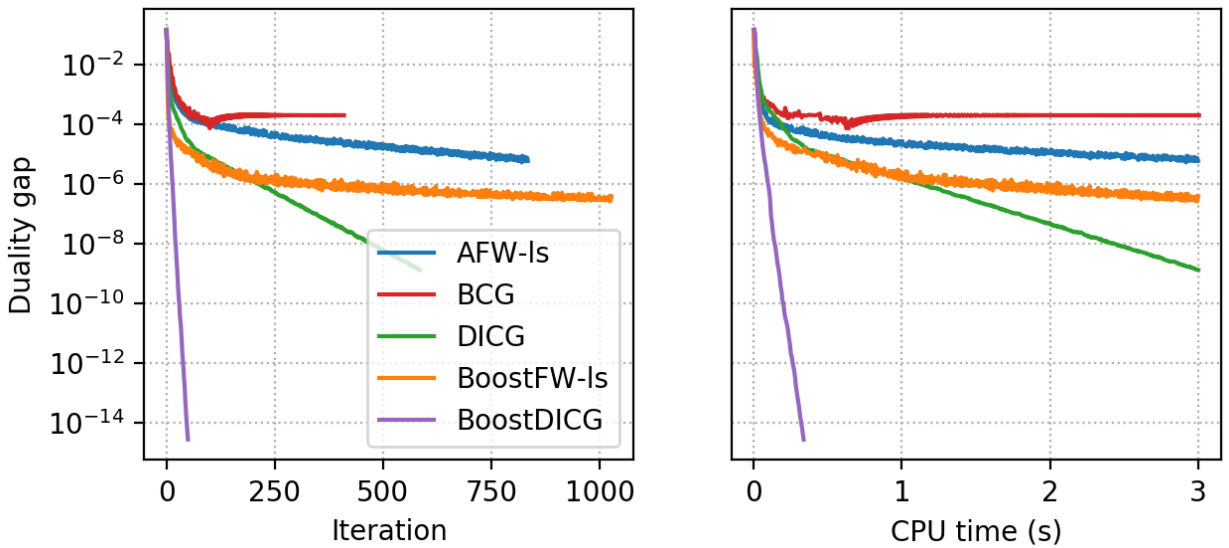


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DICG

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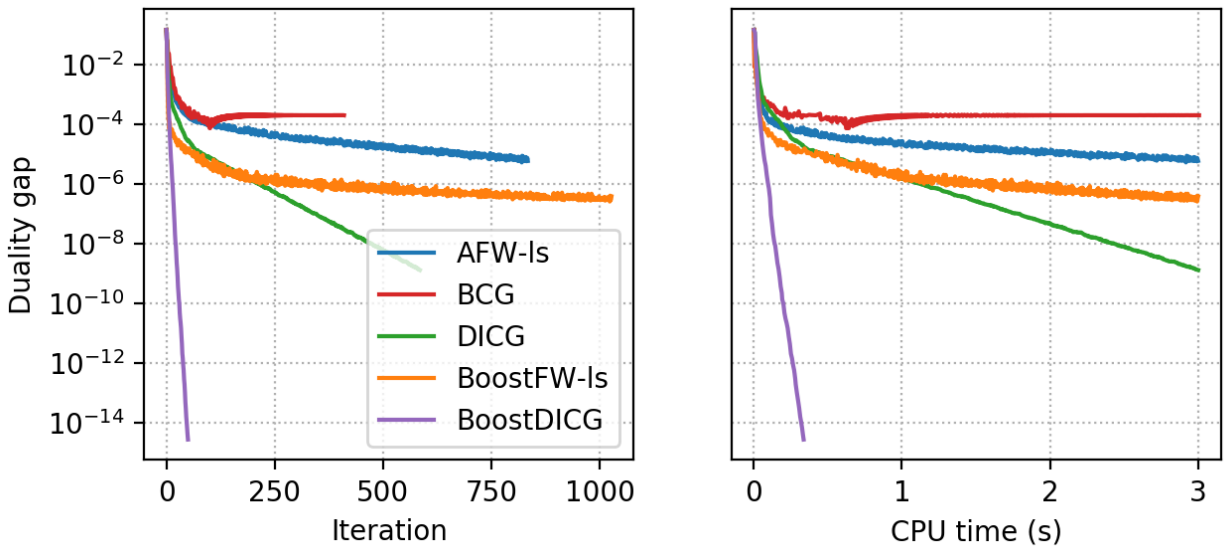


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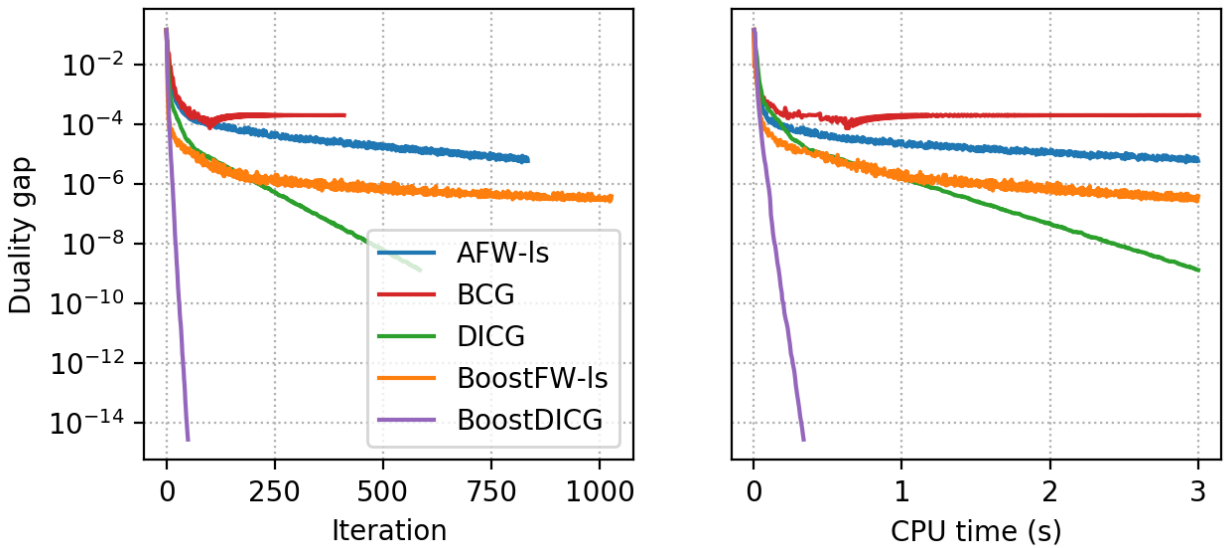


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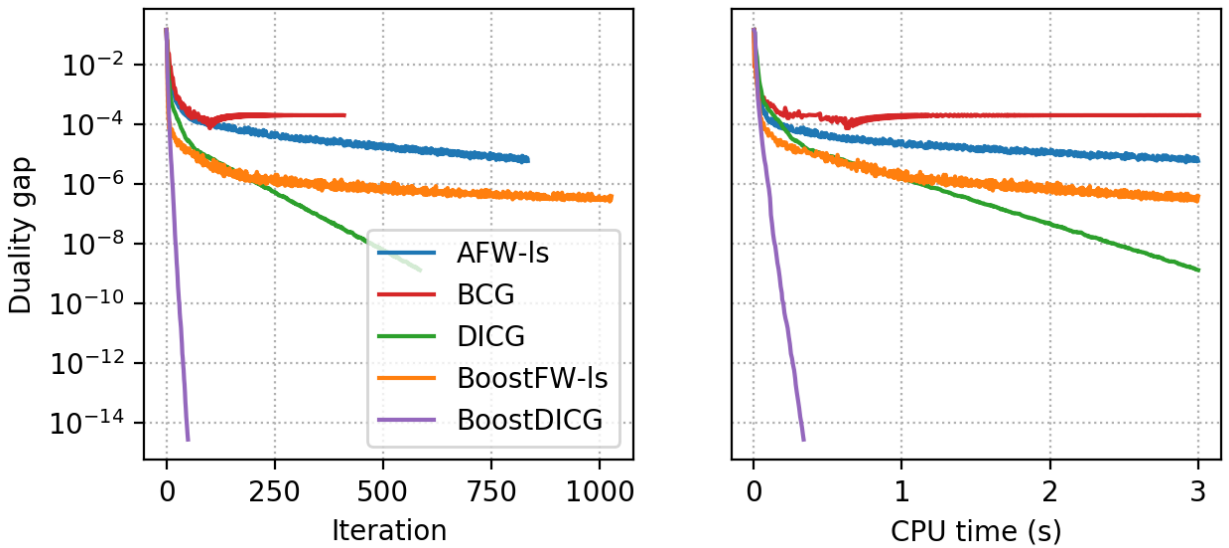


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# Large-scale optimization

Consider

$$\min \left\{ f(x) := \frac{1}{m} \sum_{i=1}^{m} f_i(x) \right\}$$
$$\text{s.t. } x \in \mathcal{C}$$

where

- $\mathcal{C} \subset \mathbb{R}^n$ is a compact convex set
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Computing $f(x)$ or $\nabla f(x)$ is too expensive

- Cannot use line search
- More efficient to use an estimator $\tilde{\nabla} f(x)$ to get approximate (but cheap) gradient information

# Stochastic Frank-Wolfe algorithms

**Template** Stochastic Frank-Wolfe

**Input:** $x_0 \in \mathcal{C}$, $\gamma_t \in [0, 1]$

1: **for** $t = 0$ **to** $T - 1$ **do**

2: Update the gradient estimator $\tilde{\nabla} f(x_t)$

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- The *vanilla* Stochastic Frank-Wolfe algorithm (SFW) estimates the gradient by averaging over a minibatch of size $b_t$:

$$\tilde{\nabla} f(x_t) \leftarrow \frac{1}{b_t} \sum_{j=1}^{b_t} \nabla f_{i_j}(x_t) \quad \text{where} \quad i_1, \ldots, i_{b_t} \overset{\text{i.i.d.}}{\sim} \mathcal{U}(\llbracket 1, m \rrbracket)$$

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- New variants require only $b_t \sim t$ or $b_t \sim 1$

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Simultaneously proposed by Duchi et al. (2011) and McMahan & Streeter (2010):

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**Algorithm** Adaptive Gradient (AdaGrad)

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$$x_{t+1} \leftarrow \arg\min_{x \in \mathcal{C}} \|x - (x_t - \eta H_t^{-1} \tilde{\nabla} f(x_t))\|_{H_t}$$

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- The offset $\delta$ prevents from dividing by zero
- The step-size automatically adjusts to the geometry of the problem
- Larger step-sizes are given to infrequent (but potentially very informative) features whenever they appear so that they do not go unnoticed

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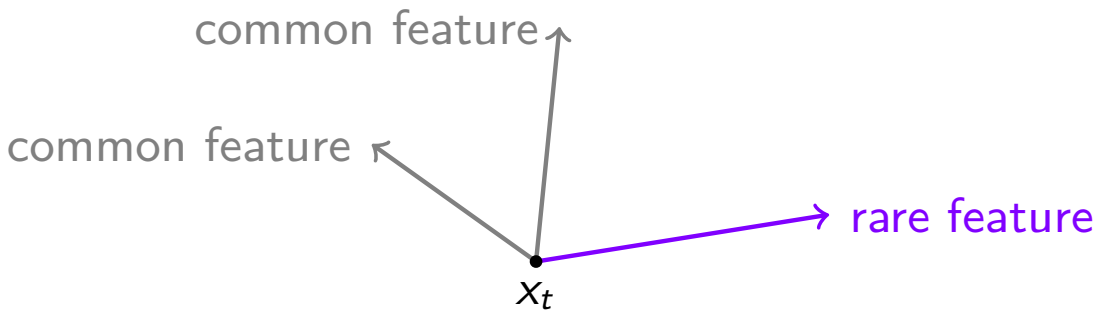

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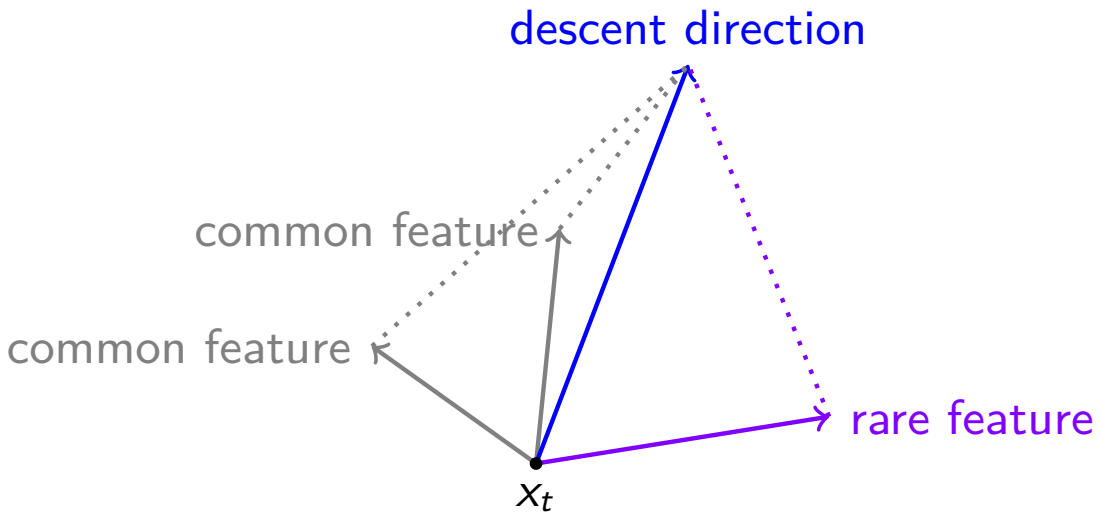

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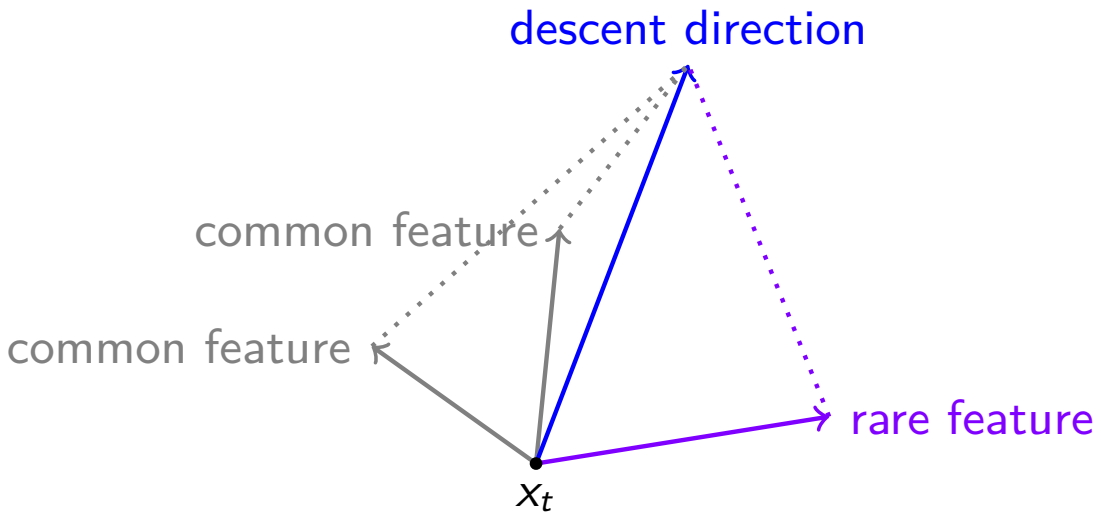


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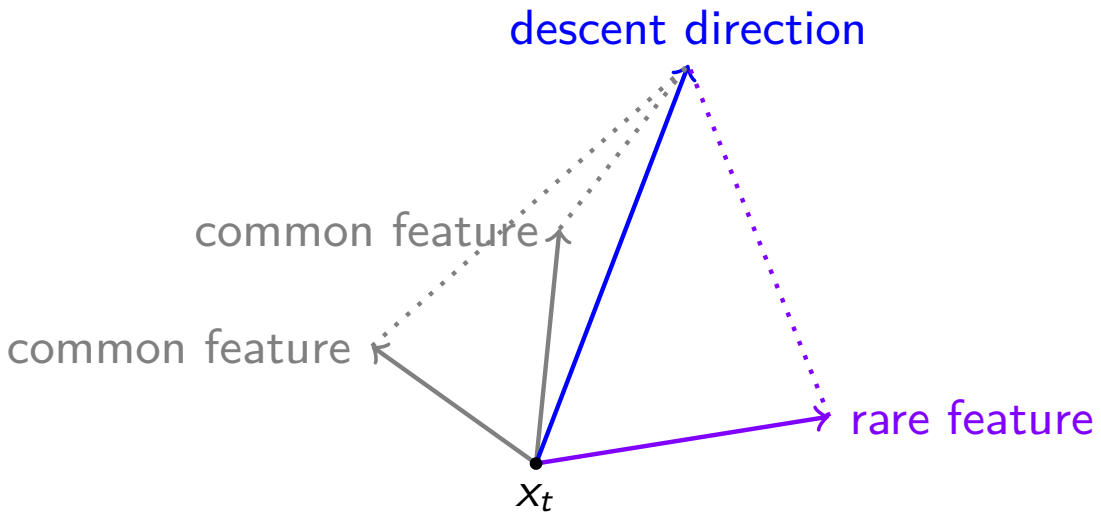


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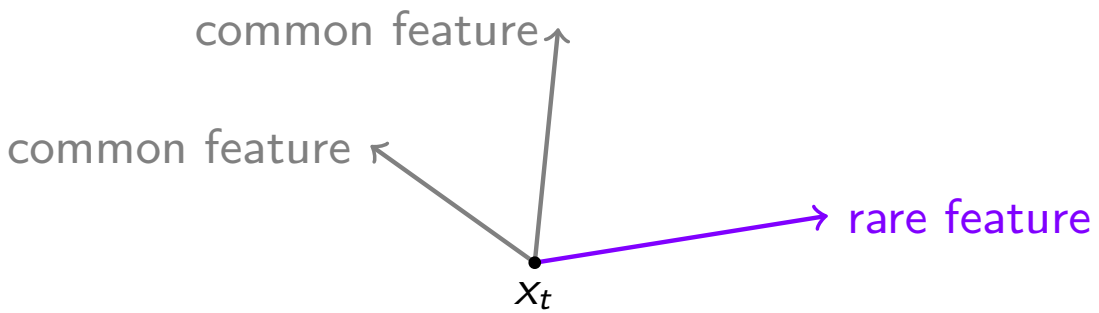

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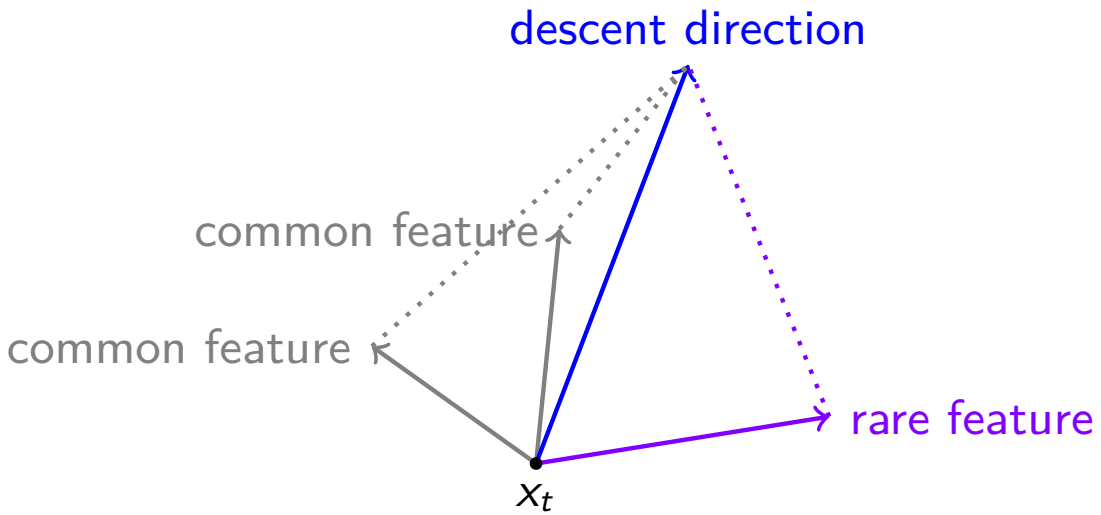


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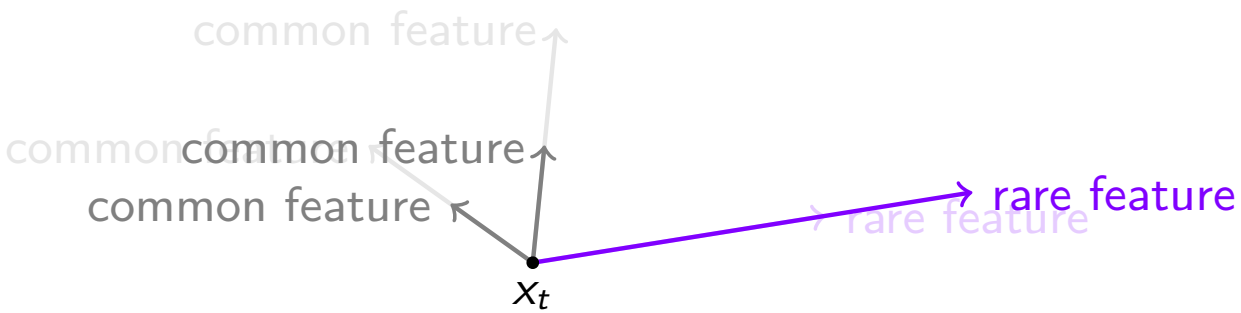

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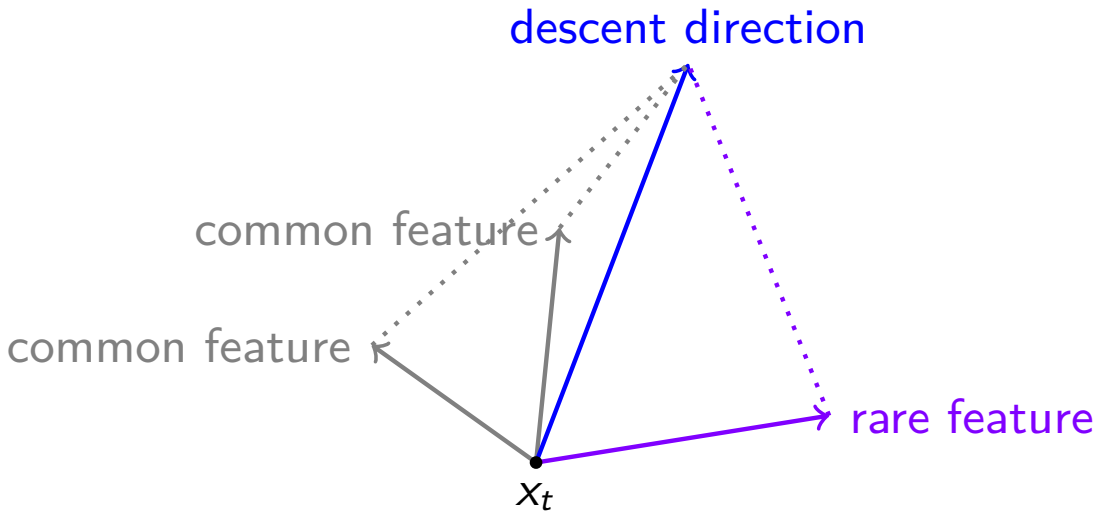


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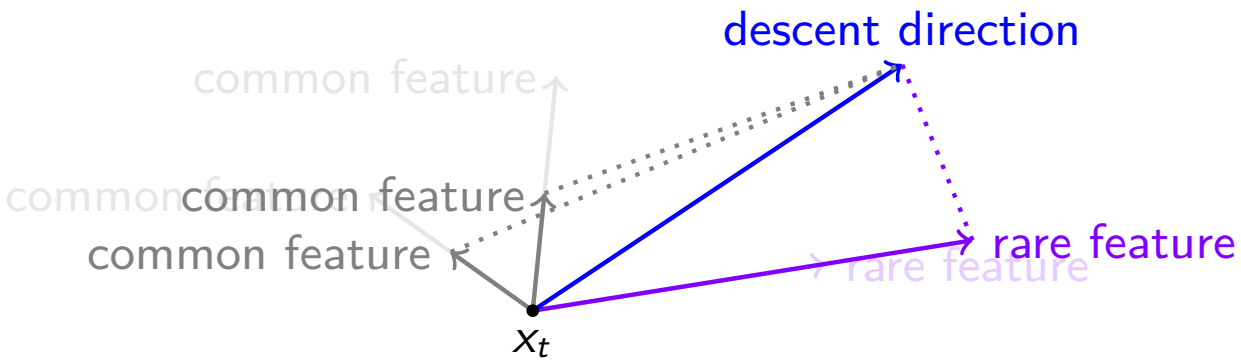

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- We would likely lose the precious properties of the descent directions of AdaGrad

# Frank-Wolfe with adaptive gradients via sliding

- Instead, consider the constrained subproblem occurring at every iteration:

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Idea:

- Solve the subproblem using FW (*sliding* technique of Lan & Zhou (2016))
- Run only a small and fixed number $K$ of iterations of FW ($K \sim 5$)
- That is, we claim that leveraging just a small amount of information from the adaptive metric $H_t$ is enough

# The method

**Template** Frank-Wolfe with adaptive gradients

**Input:** $x_0 \in \mathcal{C}$, $0 < \lambda_t^- \leqslant \lambda_{t+1}^- \leqslant \lambda_{t+1}^+ \leqslant \lambda_t^+$, $K \in \mathbb{N}\backslash\{0\}$, $\eta > 0$, $\gamma_t \in [0, 1]$

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# The method

**Template** Frank-Wolfe with adaptive gradients

**Input:** $x_0 \in \mathcal{C}$, $0 < \lambda_t^- \leqslant \lambda_{t+1}^- \leqslant \lambda_{t+1}^+ \leqslant \lambda_t^+$, $K \in \mathbb{N}\backslash\{0\}$, $\eta > 0$, $\gamma_t \in [0,1]$

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- Define AdaX depending on the strategy for $\tilde{\nabla} f(x_t)$: AdaSFW, AdaSVRF, etc.

# Convergence analysis

**Theorem**

*Let $f_1, \ldots, f_m \colon \mathbb{R}^n \to \mathbb{R}$ be $L$-smooth convex functions with $G = \max_{\mathcal{C}} \|\nabla f_i\|_2$. Then AdaSFW with $b_t \leftarrow \big(G(t+2)/(LD)\big)^2$, $\eta_t \leftarrow \lambda_t^- / L$, and $\gamma_t \leftarrow 2/(t+2)$ satisfies*

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# Computational experiments

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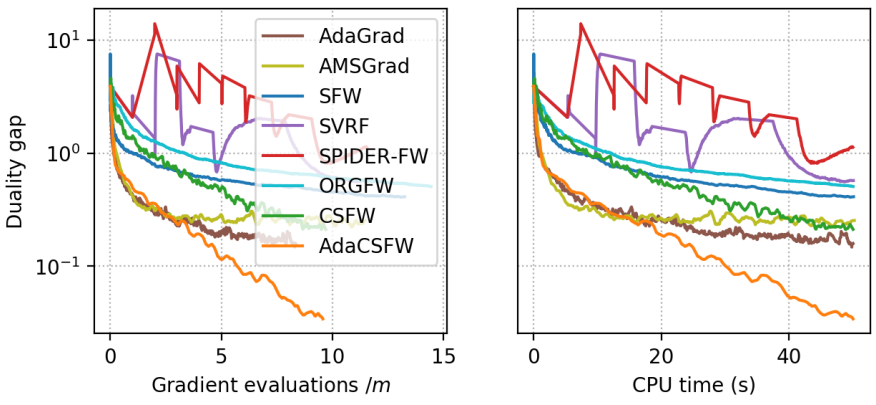
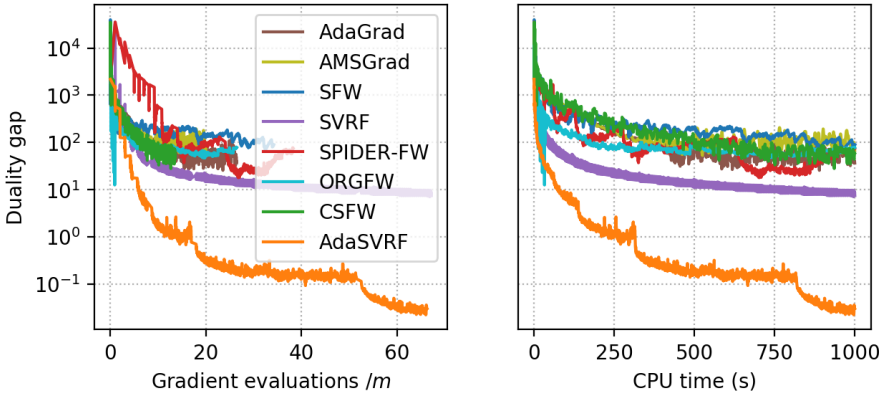
- We set $K \sim 5$

# Convex objectives

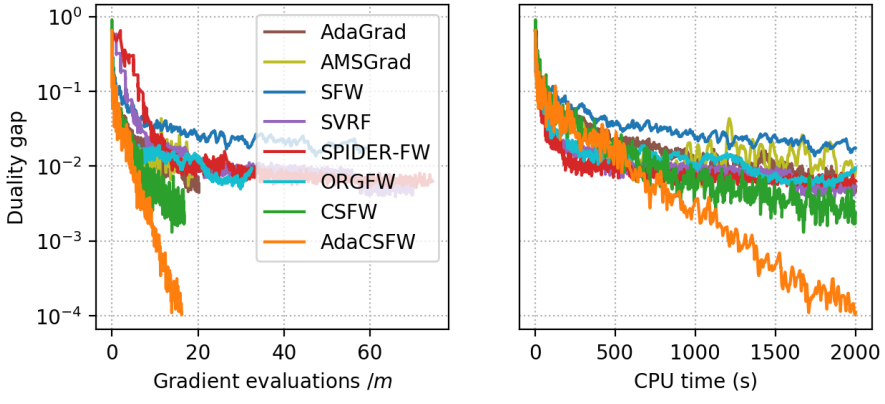
- Support vector classification on a synthetic dataset




- Linear regression on the YearPredictionMSD dataset

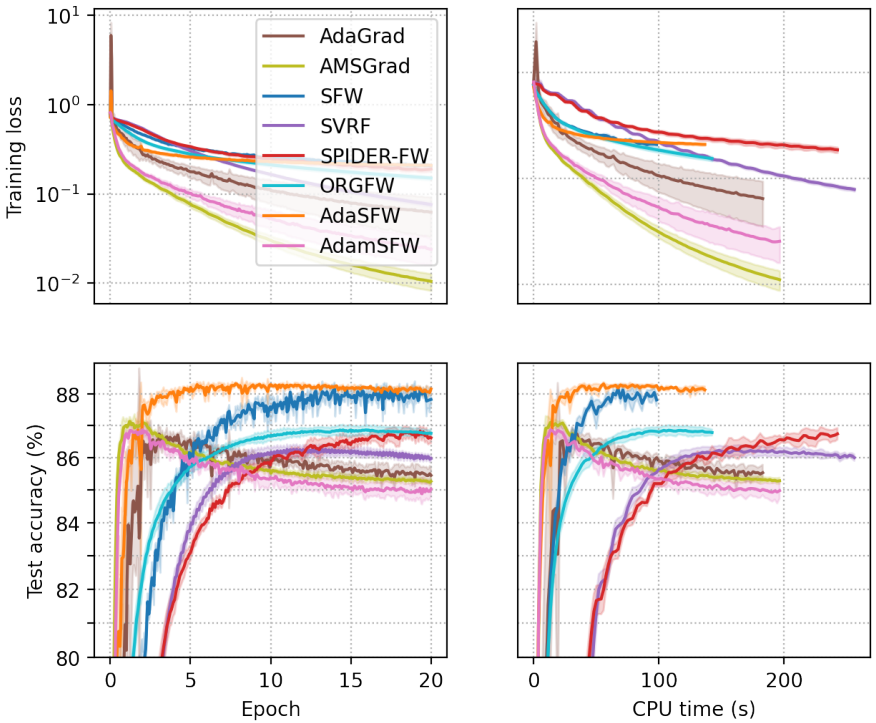



- Logistic regression on the RCV1 dataset

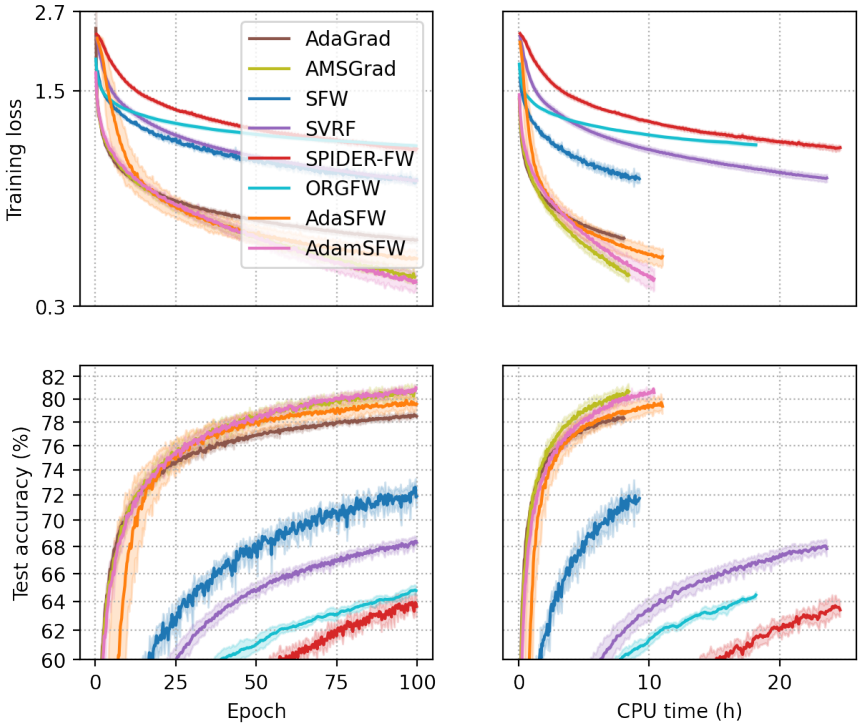


# Nonconvex objectives

- Neural network with one hidden layer on the IMDB dataset




- Convolutional network on the CIFAR-10 dataset




- Each layer is constrained into an $\ell_\infty$-ball

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**Theorem (Carathéodory, 1907)**

*Let $\mathcal{C} \subset \mathbb{R}^n$ be a compact convex set and $x^* \in \mathcal{C}$. Then $x^*$ can be represented as a convex combination of at most $n+1$ vertices of $\mathcal{C}$.*

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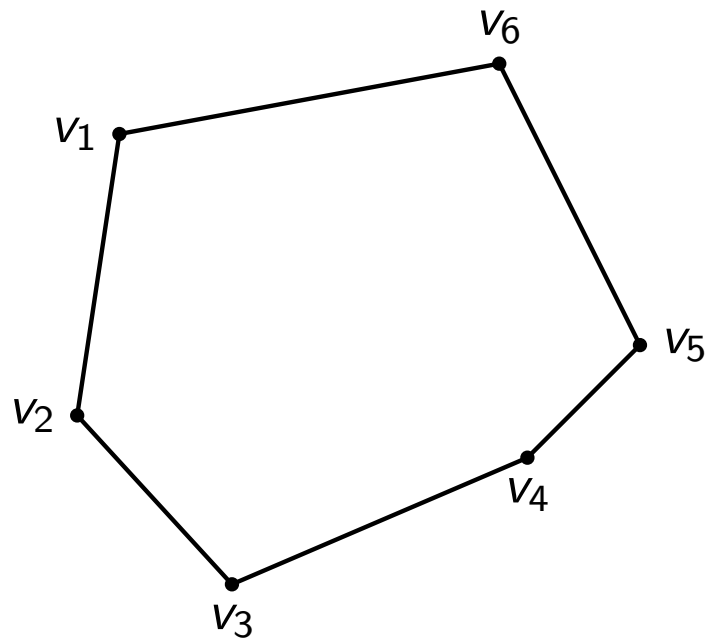

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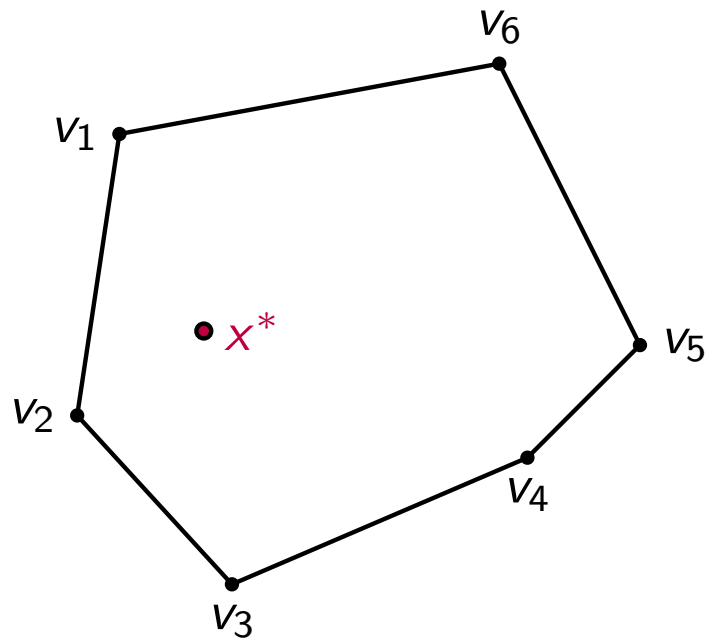

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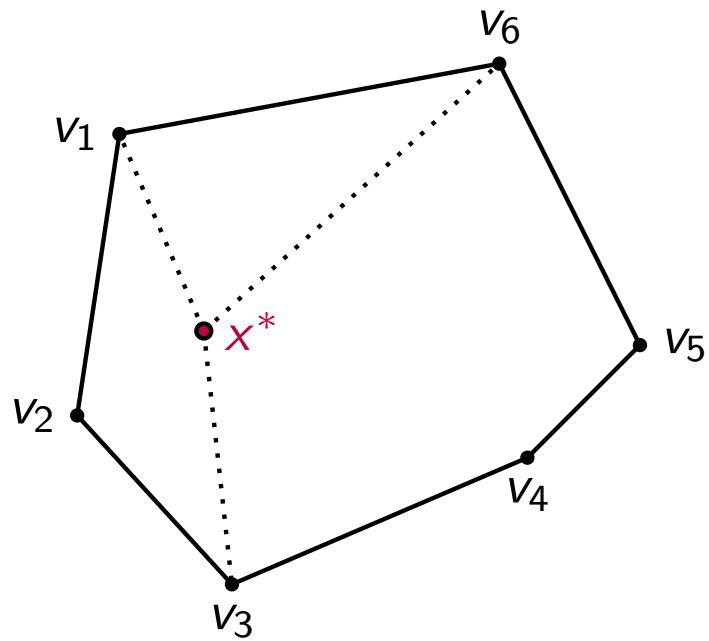

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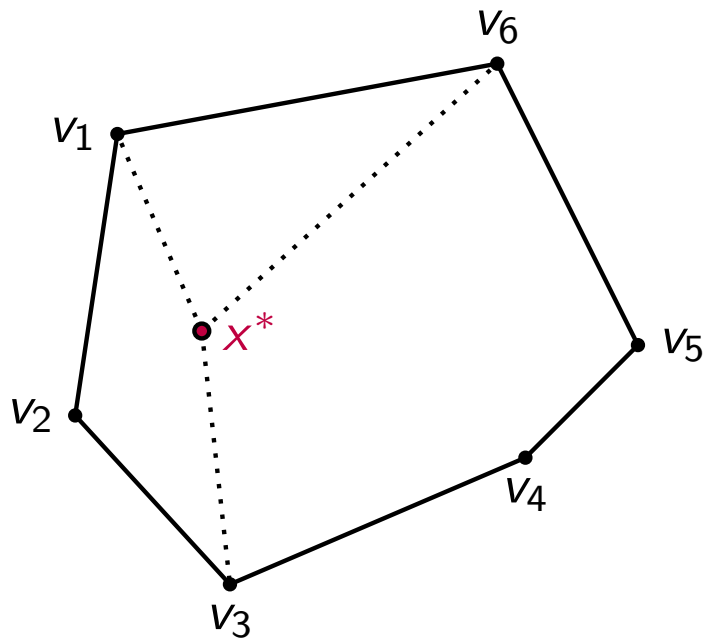

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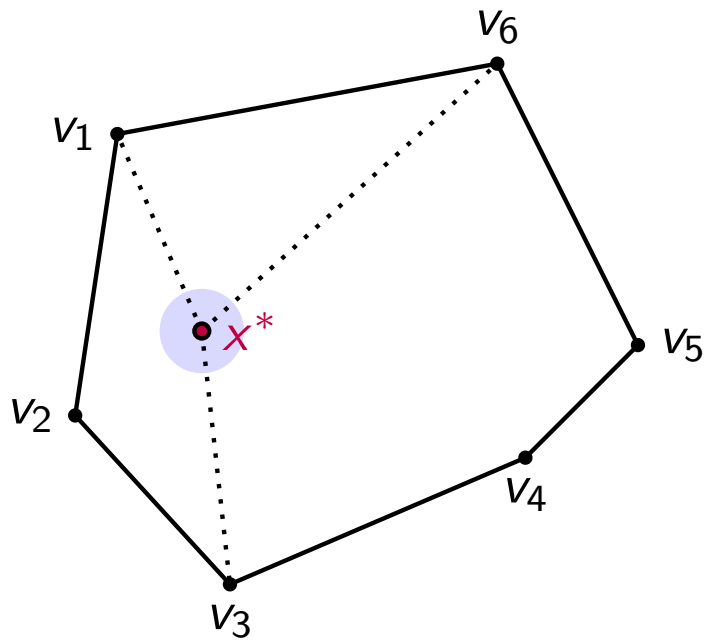

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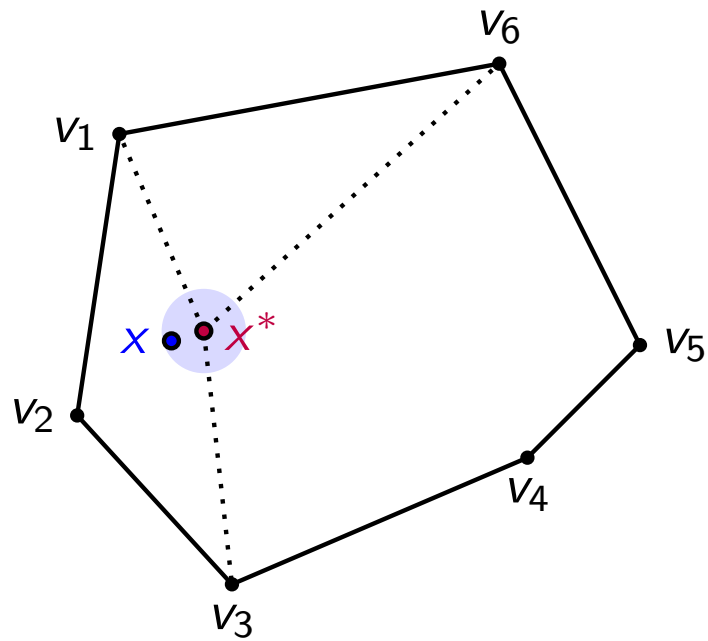

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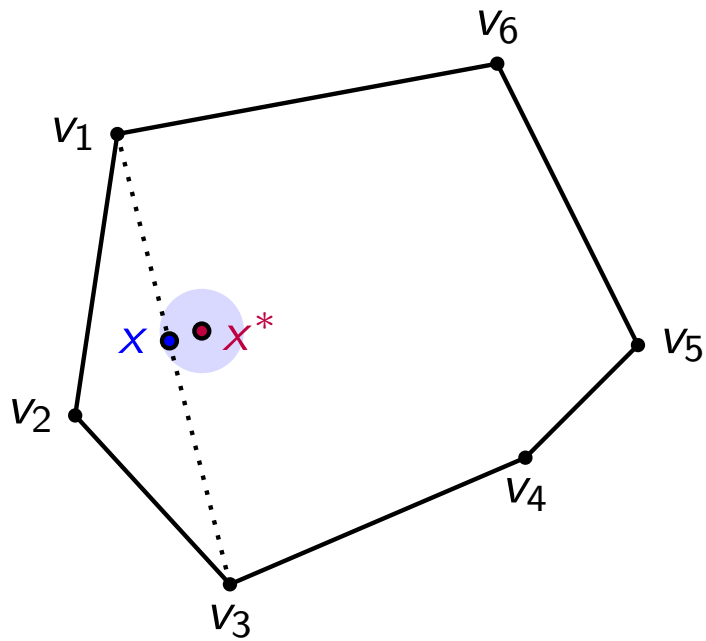

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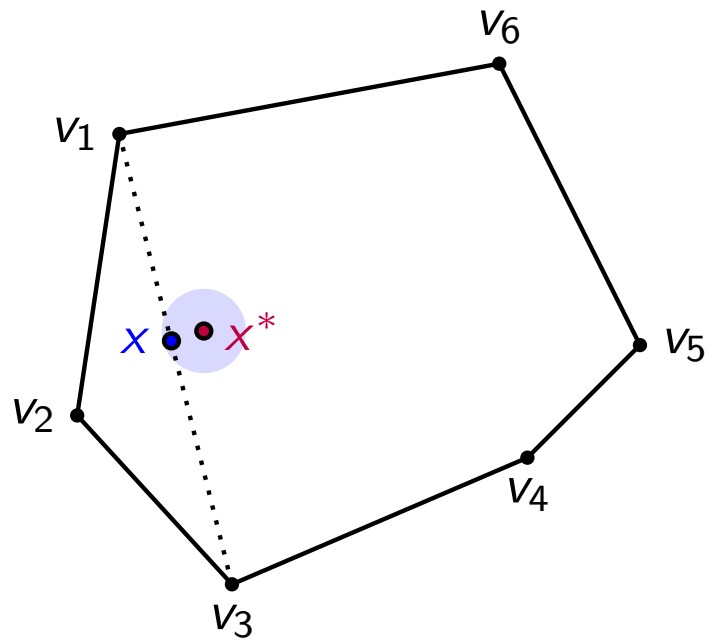
- Define the *cardinality* of $x \in \mathcal{C}$ as the number of vertices in a given convex decomposition of $x$



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**Lemma**

*Let $p \in [2, +\infty[$ and $f(x) = \frac{1}{2}\|x - x^*\|_p^2$. Then $f$ is convex, $(p-1)$-smooth, and 1-gradient dominated w.r.t. the $\ell_p$-norm.*

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**Algorithm** Hybrid Conditional Gradient-Smoothing (HCGS)

**Input:** $x_0 \in \mathcal{C}$, $G_2, D_2$

1: **for** $t = 0$ **to** $T - 1$ **do**
2: $\quad \beta_t \leftarrow 2(D_2/G_2)/\sqrt{t+2}$
3: $\quad v_t \leftarrow \arg\min_{v \in \mathcal{C}} \langle v, \nabla f_{\beta_t}(x_t) \rangle$
4: $\quad \gamma_t \leftarrow 2/(t+2)$
5: $\quad x_{t+1} \leftarrow x_t + \gamma_t(v_t - x_t)$

# The Hybrid Conditional Gradient-Smoothing algorithm

Smoothen the problem (Argyriou et al., 2014) by taking the Moreau envelope (Moreau, 1965):

$$f_\beta(x) = \min_{y \in \mathbb{R}^n} f(y) + \frac{1}{2\beta}\|x - y\|_2^2 \quad \text{and} \quad \nabla f_\beta(x) = \frac{1}{\beta}(x - \text{prox}_{\beta f}(x))$$

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5: $\quad x_{t+1} \leftarrow x_t + \gamma_t(v_t - x_t)$

**Lemma (Argyriou et al., 2014)**

*For all $\beta \geqslant \beta' > 0$, $f_\beta \leqslant f \leqslant f_\beta + \beta G_2^2/2$ and $f_\beta \leqslant f_{\beta'} \leqslant f_\beta + (\beta - \beta')G_2^2/2$.*

# The Hybrid Conditional Gradient-Smoothing algorithm

**Theorem (Argyriou et al., 2014)**

*Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex $G_2$-Lipschitz continuous function w.r.t. the $\ell_2$-norm. Then*

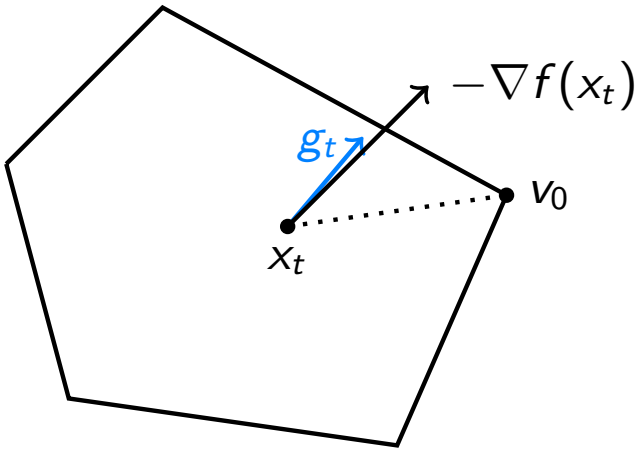
$$f(x_t) - \min_{\mathcal{C}} f \leqslant \frac{4 G_2 D_2}{\sqrt{t+1}}.$$

# Cardinality bounds

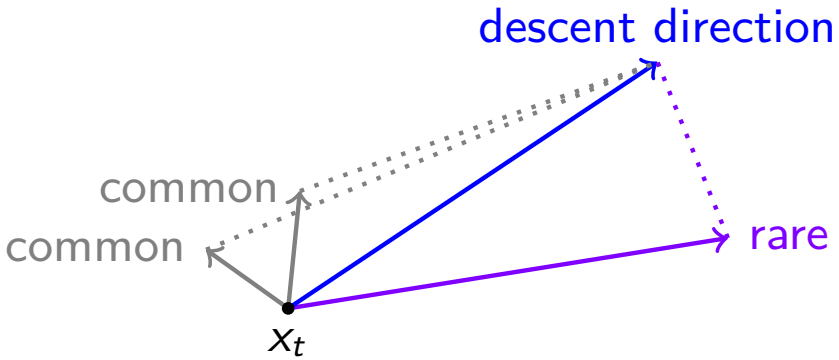
| $p$ | **Assumption** | **Cardinality bound** | |
|---|---|---|---|
| | | Via Frank-Wolfe | Related work |
| $[2, +\infty[$ | – | $\mathcal{O}\left(\frac{pD_p^2}{\varepsilon^2}\right)$ | $\mathcal{O}\left(\frac{pD_p^2}{\varepsilon^2}\right)$ |
| | $x^* \in \text{int}(\mathcal{C})$ | $\mathcal{O}\left(p\left(\frac{D_p}{r_p}\right)^2 \ln\left(\frac{1}{\varepsilon}\right)\right)$ | $\mathcal{O}\left(p\left(\frac{D_p}{r_p}\right)^2 \ln\left(\frac{1}{\varepsilon}\right)\right)$ |
| | $\mathcal{C}$ strongly convex | $\mathcal{O}\left(\frac{\sqrt{p}D_p + p/\alpha_p}{\varepsilon}\right)$ | – |
| $]1, 2[$ | – | $\mathcal{O}\left(\frac{n^{(2-p)/p}D_2^2}{\varepsilon^2}\right)$ | $\mathcal{O}\left(\frac{D_p^{p/(p-1)}}{p^{1/(p-1)}\varepsilon^{p/(p-1)}}\right)$ |
| $1$ | – | $\mathcal{O}\left(\frac{nD_2^2}{\varepsilon^2}\right)$ | – |
| $+\infty$ | – | $\mathcal{O}\left(\frac{D_2^2}{\varepsilon^2}\right)$ | $\mathcal{O}\left(\frac{\ln(n)D_\infty^2}{\varepsilon^2}\right)$ |

# Conclusion

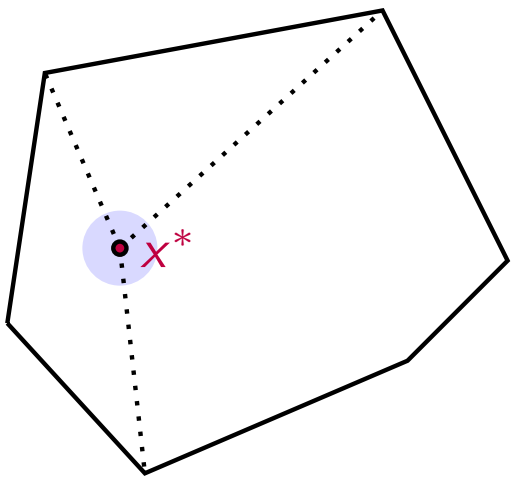
**Boosted Frank-Wolfe**




**Frank-Wolfe & adaptive gradients**




**Approximate Carathéodory**



# References (1/4)


A. Argyriou, M. Signoretto, and J. A. K. Suykens. Hybrid conditional gradient-smoothing algorithms with applications to sparse and low rank regularization. Chapman & Hall/CRC, 2014

S. Barman. Approximating Nash equilibria and dense bipartite subgraphs via an approximate version of Carathéodory's theorem. *STOC*, 2015

G. Braun, S. Pokutta, D. Tu, and S. Wright. Blended conditional gradients: the unconditioning of conditional gradients. *ICML*, 2019

M. D. Canon and C. D. Cullum. A tight upper bound on the rate of convergence of Frank-Wolfe algorithm. *SIAM J. Control*, 1968

**C. W. Combettes and S. Pokutta. Blended matching pursuit. *NeurIPS*, 2019**

**C. W. Combettes and S. Pokutta. Revisiting the approximate Carathéodory problem via the Frank-Wolfe algorithm. Minor revision at *Math. Program.*, 2021**

**C. W. Combettes and S. Pokutta. Boosting Frank-Wolfe by chasing gradients. *ICML*, 2020**

**C. W. Combettes and S. Pokutta. Complexity of linear minimization and projection on some sets. Minor revision at *Oper. Res. Lett.*, 2021**

**C. W. Combettes, C. Spiegel, and S. Pokutta. Projection-free adaptive gradients for large-scale optimization. *Submitted*, 2021**

L. Condat. Fast projection onto the simplex and the $\ell_1$ ball. *Math. Program.*, 2016

# References (2/4)


V. F. Demyanov and A. M. Rubinov. *Approximate Methods in Optimization Problems*. Elsevier, 1970

J. C. Duchi, E. Hazan, and Y. Singer. Adaptive subgradient methods for online learning and stochastic optimization. *J. Mach. Learn. Res.*, 2011

J. C. Dunn and S. Harshbarger. Conditional gradient algorithms with open loop step size rules. *J. Math. Anal. Appl.*, 1978

M. Frank and P. Wolfe. An algorithm for quadratic programming. *Naval Res. Logist. Q.*, 1956

D. Garber and O. Meshi. Linear-memory and decomposition-invariant linearly convergent conditional gradient algorithm for structured polytopes. *NIPS*, 2016

E. Hazan and H. Luo. Variance-reduced and projection-free stochastic optimization. *ICML*, 2016

J. Guélat and P. Marcotte. Some comments on Wolfe's 'away step'. *Math. Program.*, 1986

Y. Haugazeau. *Sur les Inéquations Variationnelles et la Minimisation de Fonctionnelles Convexes*. Thèse de doctorat, Univ. Paris, 1968

M. Jaggi. Revisiting Frank-Wolfe: Projection-free sparse convex optimization. *ICML*, 2013

D. P. Kingma and J. Ba. Adam: A method for stochastic optimization. *ICLR*, 2015

S. Lacoste-Julien and M. Jaggi. On the global linear convergence of Frank-Wolfe optimization variants. *NIPS*, 2015

# References (3/4)


G. Lan. The complexity of large-scale convex programming under a linear optimization oracle. *arXiv*, 2013

G. Lan and Y. Zhou. Conditional gradient sliding for convex optimization. *SIAM J. Optim.*, 2016

E. S. Levitin and B. T. Polyak. Constrained minimization methods. *USSR Comp. Math. Math. Phys.*, 1966

F. Locatello, M. Tschannen, G. Rätsch, and M. Jaggi. Greedy algorithms for cone constrained optimization with convergence guarantees. *NIPS*, 2017

H. B. McMahan and M. Streeter. Adaptive bound optimization for online convex optimization. *COLT*, 2010

V. Mirrokni, R. Paes Leme, A. Vladu, and S. C.-W. Wong. Tight bounds for approximate Carathéodory and beyond. *ICML*, 2017

J. J. Moreau. Proximité et dualité dans un espace hilbertien. *Bull. Soc. Math. France*, 1965

G. Négiar, G. Dresdner, A. Y.-T. Tsai, L. El Ghaoui, F. Locatello, R. M. Freund, and F. Pedregosa. Stochastic Frank-Wolfe for constrained finite-sum minimization. *ICML*, 2020

A. S. Nemirovsky and D. B. Yudin. *Problem Complexity and Method Efficiency in Optimization*. Wiley, 1983

G. Pisier. Remarques sur un résultat non publié de B. Maurey. *Ec. polytech.*, 1981

S. J. Reddi, S. Kale, and S. Kumar. On the convergence of Adam and beyond. *ICLR*, 2018

# References (4/4)


Z. Shen, C. Fang, P. Zhao, J. Huang, and H. Qian. Complexities in projection-free stochastic non-convex minimization. *AISTATS*, 2019

P. Wolfe. Convergence theory in nonlinear programming. *Integer and Nonlinear Programming*. North-Holland, 1970

J. Xie, Z. Shen, C. Zhang, H. Qian, and B. Wang. Efficient projection-free online methods with stochastic recursive gradient. *AAAI*, 2020

A. Yurtsever, S. Sra, and V. Cevher. Conditional gradient methods via stochastic path-integrated differential estimator. *ICML*, 2019

# Stochastic Frank-Wolfe algorithms

| **Algorithm** | **Update** $\tilde{\nabla} f(x_t)$ |
|---|---|
| SFW | $\frac{1}{b_t}\sum_{j=1}^{b_t} \nabla f_{i_j}(x_t)$ |
| SVRF | $\nabla f(\tilde{x}_t) + \frac{1}{b_t}\sum_{j=1}^{b_t}(\nabla f_{i_j}(x_t) - \nabla f_{i_j}(\tilde{x}_t))$ |
| SPIDER-FW | $\nabla f(\tilde{x}_t) + \frac{1}{b_t}\sum_{j=1}^{b_t}(\nabla f_{i_j}(x_t) - \nabla f_{i_j}(x_{t-1}))$ |
| ORGFW | $\frac{1}{b_t}\sum_{j=1}^{b_t} \nabla f_{i_j}(x_t) + (1-\rho_t)\left(\tilde{\nabla} f(x_{t-1}) - \frac{1}{b_t}\sum_{j=1}^{b_t} \nabla f_{i_j}(x_{t-1})\right)$ |
| CSFW | $\tilde{\nabla} f(x_{t-1}) + \sum_{j=1}^{b_t}\left(\frac{1}{m} f'_{i_j}(\langle a_{i_j}, x_t\rangle) - [\alpha_{t-1}]_{i_j}\right) a_{i_j}$ |
| | and $[\alpha_t]_{i_j} \leftarrow \begin{cases} (1/m) f'_{i_j}(\langle a_{i_j}, x_t\rangle) & \text{if } i_j \in \{i_1, \ldots, i_{b_t}\} \\ [\alpha_{t-1}]_{i_j} & \text{else} \end{cases}$ |

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# A lower bound when $p \in [2, +\infty[$

**Theorem (Mirrokni et al., 2017)**

*Let $p \in [2, +\infty[$, $H_n \in \mathbb{R}^{n \times n}$ be the Hadamard matrix of dimension $n$, $\mathcal{C} = \mathrm{conv}(H_n / n^{1/p})$ be the convex hull of the $\ell_p$-normalized columns of $H_n$, and $x^* = e_1 / n^{1/p} \in \mathcal{C}$. Then for all $x \in \mathcal{C}$,*

$$\|x - x^*\|_p \leqslant \varepsilon \Rightarrow \mathrm{card}(x) \geqslant \frac{1}{\varepsilon^2 + 1/n}.$$

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- Sylvester's construction:

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gives

$$H_1 = (1), \quad H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad H_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

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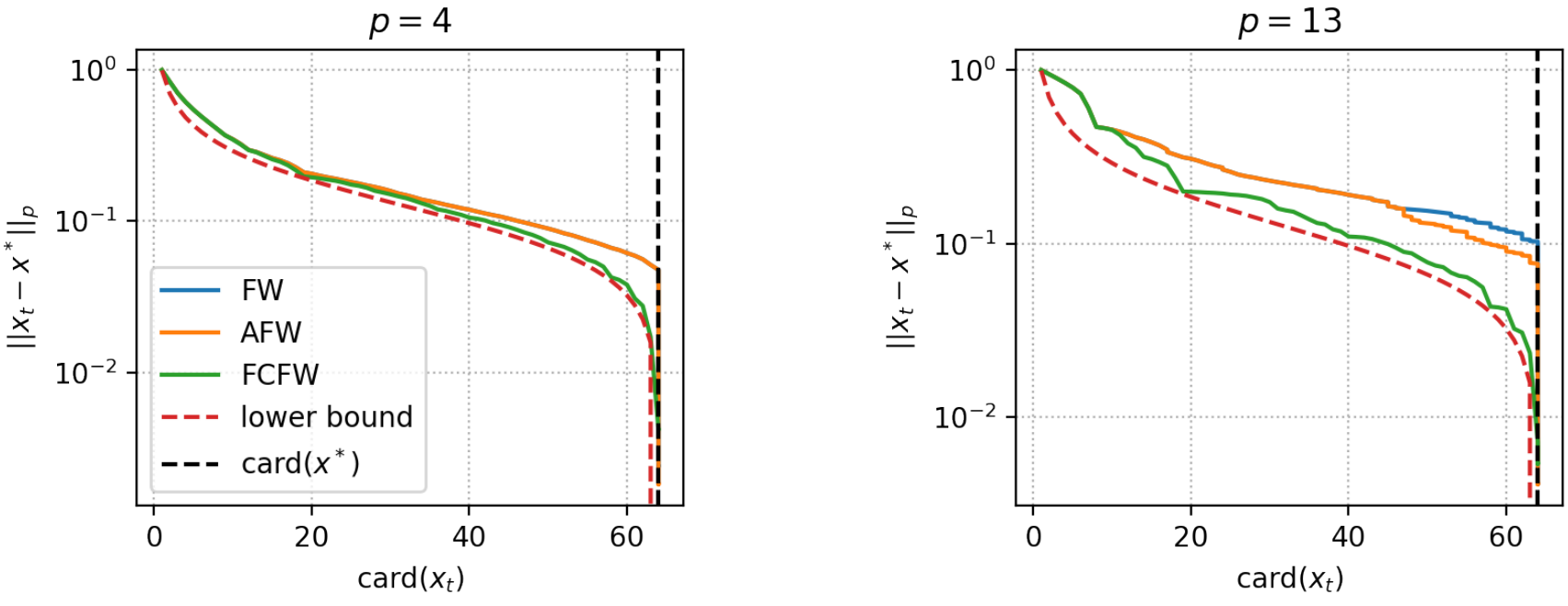


- FCFW almost matches the lower bound

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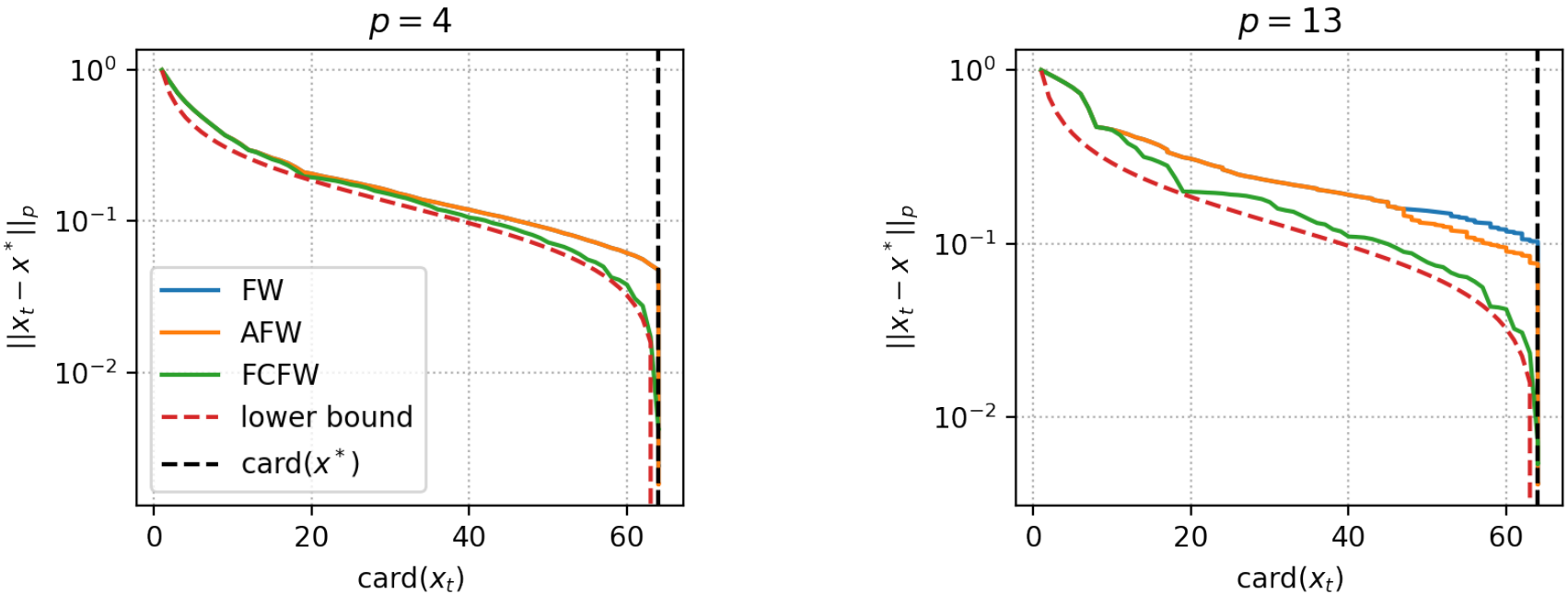


- FCFW almost matches the lower bound
- There is no precise analysis of FCFW: the current analysis is transferred from that of AFW (Lacoste-Julien & Jaggi, 2015) and holds only for smooth strongly convex functions